A DISTANCE TO THE HYADES USING A LEAST SQUARES WITH PROBABILISTIC CONSTRAINTS

By

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To my parents and to my dear friend Melody.

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Abstract of Dissertation Presented to the Graduate School
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Due to its proximity, the Hyades open cluster plays a vital role in the establishment of the galactic and extragalactic distance scales, as many of the distance indicators used to estimate the distances to other galaxies are calibrated using stars which are members of the Hyades. This importance of the Hyades stars as calibrators demands that their distances be determined as accurately as possible. Past investigations have made use of convergent point approaches, which impose the physically-ridiculous requirement that all stars which are members of the cluster must share exactly the same velocity. In addition, these investigations treated the proper motion and radial velocity data in a piecemeal manner, and have totally ignored the information provided by the trigonometric parallaxes.

For the first time, the method of least squares with probabilistic constraints has been applied to the Hyades cluster. This technique constrains the stars' velocities to obey a univariate Gaussian (i.e., Maxwellian) distribution with a dispersion of ~0.25 km s^{-1}, which seems to closely approximate the actual velocity distribution of the Hyades, and also restricts the stellar parallaxes to prevent a distortion of the cluster along the line of sight. All kinematic observations — proper motions, radial velocities and trigonometric parallaxes — can be simultaneously incorporated into a single adjustment which directly yields accurate estimates of the parallaxes and velocities of the Hyades stars.

Application of the technique to model clusters shows that the least squares with probabilistic constraints does indeed yield more accurate distances than those obtained by the classical convergent point. The models also indicate that the parameter estimates are not terribly sensitive to the form of the cluster velocity distribution, the results being similar for both Gaussian and uniform spatial and velocity distributions. Finally, an adjustment incorporating observations for 140 known members of the Hyades yields a cluster distance of 49.7 ± 1.0 parsecs, in good agreement with recent determinations by other investigators.

CHAPTER 1
INTRODUCTION

The complexity of astrometric instrumentation has increased greatly since the turn of the century. Visual observations with long focus telescopes and transit circles have been largely replaced by photographic measurements obtained by automated measuring engines, and more recently, by ground and space-based observations made with photoelectric devices and CCDs. These improved instruments have made possible milli-arcsecond precision (and, we hope, a corresponding improvement in accuracy) in proper motion and parallax observations, an order of magnitude better than the best observations available in the early 1900's. An even greater improvement has been made in the precision of radial velocity determinations. Spectrometers placed at the coudé focus of large reflectors can now produce radial velocities with a precision corresponding to a standard deviation of 0.25 km s^{-1} (Griffin et al., 1988), a decrease in standard deviation by a factor of more than 20 as compared to spectrographic observations.

A similar sort of advancement has occurred in the mathematics of data adjustment, which has its origins in the techniques developed by 19th century astronomers to cope with the problems involved in orbit determination. Mathematicians have spent great amounts of time laboring to generalize and extend these techniques, especially with regard to the method of least squares, the adjustment procedure most commonly used. Unfortunately, these more recent developments have been largely ignored by astronomers, the vast majority of whom still employ (often erroneously) the rather restrictive "classical" least squares in their data adjustments[1]. The motivation behind the construction of more precise instrumentation and the development of general, flexible adjustment procedures is the same, namely, to obtain accurate estimates of certain parameters, and the use of both is necessary to obtain the most accurate possible parameter estimates. Today's astronomers spend huge amounts of time building and utilizing marvelously sophisticated instruments, only to then reduce the data by numerical techniques developed long before the introduction of computers.

Several circumstances may help to explain this state of affairs. The first is that the typical astronomer is neither a mathematician nor a statistician; most have little interest in expending the time required to learn new adjustment techniques beyond those learned while they were still students, especially if those techniques are complex. Such interest, where originally present, is sometimes extinguished by the language and notation of the mathematicians, which can be obscure and confusing to a scientist, especially if there are no examples of the technique's implementation. Another reason, perhaps the most prevalent, is that more general adjustment techniques often yield parameters little different from those obtained via classical least squares. Lack of fluency in programming languages, unfamiliarity with sophisticated modern computers and the considerable time required to produce complex programs are obstacles faced by many astronomers in implementing modern adjustment techniques. It is usually far easier to make use of one of the many readily available classical least-squares adjustment routines, thereby reducing programming time and the number of headaches. Eichhorn summarizes the situation quite nicely:

> Unfortunately, most users of the least-squares adjustment technique regard it as a mere trade tool, which comes pre-packaged and in various degrees of sophistication. If the tool is complicated, learning how to use it may consume a lot of time and in addition be difficult, especially when — as happens not infrequently — the author of

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[1] In "classical" least squares only one of the observations (which are assumed to be uncorrelated) appears in each condition equation, which must be linear.

> the instructions for using the tool is less than perfectly lucid. One also encounters often the — much wiser than immediately apparent — attitude that "if it ain't broken, don't fix it!", which produces a reluctance in much of the community of users to abandon a tried-and-true, even though not optimally accurate adjustment technique. . . (Eichhorn, 1993, p. 338)

Such is the case with past determinations of the distances to the stars which are considered to be physical members of the Hyades star cluster. This cluster's close proximity and importance as a calibrator in the galactic and extragalactic distance scales have stimulated many kinematic investigations, beginning with that of Boss (1908). The precision of the observations has improved, but alas, the sophistication of the adjustment techniques used to determine the distances to the member stars has not. All past determinations have neglected to incorporate the various types of available kinematic data (proper motions, radial velocities and parallaxes) simultaneously into a single adjustment, opting instead to treat the data in a piecemeal fashion. Indeed, the observed parallaxes, if considered at all, are simply averaged using some sort of weighting scheme and used as a "check" on the results of the adjustment procedure in which the parallax has been eliminated from the condition equations. Further, the nonlinear equations of condition are often linearized in such a manner as to introduce systematic errors (i.e., biases) in the estimated parameters, especially in the investigations performed in the first decades of this century. There was no realistic alternative to using such methodology in investigations conducted before high speed computers became readily accessible, as an adjustment involving observations of a few hundred stars and multiple equations of condition can strain the capabilities of even today's computers. Yet, the most recent published investigations still fail even to incorporate the parallax observations into the adjustments.

Previous kinematic distance determinations have assumed that all stars belonging to the Hyades cluster move with exactly the same velocity, which makes no physical sense. As Hanson (1975) points out, the "principle would be inapplicable in the case of the Hyades if . . . *random* motions of individual members contributed significantly to the stars' space velocities" (p. 393). The assumption of a common space motion has yielded acceptable results only because open clusters like the Hyades are loosely bound, with velocity components dispersed on the order of 0.5 km s^{-1}. This is about 1% of the Hyades space speed, which is approximately 44 km s^{-1}. Yet, adjustment techniques exist that would allow the individual stellar velocities to vary according to a Gaussian distribution (Eichhorn, 1977). While it is probable that the velocity distribution function of the Hyades stars may not be Gaussian in nature, an approach of this sort is more physically realistic than imposing the physically impossible and rigorous constraint that all stars share the same velocity.

These same investigations determined an individual stellar parallax by placing the star at a distance such that the space velocity of the star corresponded to its observed proper motion. However, the unavoidable errors in the proper motions will thus cause the cluster to appear more elongated along the line-of-sight direction than it actually is. Again, more accurate estimates of the stellar parallaxes may be obtained by assuming a spherical shape for the cluster and stochastically constraining the stars to lie within these bounds (Eichhorn, 1977).

The above paragraphs summarize the stimuli for the work presented in this paper, which has two primary purposes: The first of these is the goal of any adjustment, namely, to obtain from the observed data the most accurate available estimates of the unknown parameters, which in this case are the velocities and parallaxes of the Hyades stars. The second, and more important aim, is to break with the past and apply for the first time a modern, general least-squares adjustment technique to the Hyades kinematic observations. It is hoped that this investigation will be a step toward altering the perceptions of the astronomical community regarding general least-squares adjustments. Like modern instruments, these techniques are complex and sometimes problematic, but only they can fully utilize the existing data to produce the most accurate results. Today's observations deserve no lesser treatment.

CHAPTER 2
PREVIOUS HYADES DISTANCE DETERMINATIONS

Overview

Past distance determinations to the Hyades stars may be divided into two broad groups: Those making use of geometric-kinematic data (proper motions, radial velocities and parallaxes) and those obtained by astrophysical considerations (photometric "parallaxes", stellar models, the Wilson-Bappu effect, etc.). An examination of the literature also suggests that the date of an investigation generally places it into one of three distinct "epochs," each characterized by a particular school of thought and use of adjustment procedures. These epochs are summarized in the paragraphs below.

Historic (1908-1945). This epoch commences with the pioneering convergent point solution by Boss (1908), followed in short order by investigations conducted by such notables as Plummer (1912), Rasmuson (1921), and Smart (1939). These investigations are characterized by convergent point solutions using Charlier's method (Charlier, 1916) or a variant thereof, and utilize proper motions on the system of the General Catalogue (GC) with sparse radial velocities of (by contemporary standards) poor precision. The distance to the Hyades[1] was determined to be under 40 parsecs, a representative value being that of 36.3 ± 1.2 parsecs[2] obtained by Smart. This period was brought to an abrupt end by Frederick Seares (1945), who pointed out that the coefficients of the unknown parameters in the Charlier equations of condition involved the proper motion observations and that the errors in these observations would introduce a bias into the convergent point solution. He developed a new set of normal equations which were applied to Smart's material, resulting in a cluster distance of 39.2 ± 1.2 parsecs, a change of some 7%.

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[1] By "the distance to the Hyades", the author refers to the distance of the geometrical center of the cluster as determined by the stars considered in the adjustment.

[2] The numbers given after the ± are standard errors, unless otherwise indicated.

Classic (1950-1966). This epoch may also be referred to as the "heyday of the convergent point," for it is during this period that the "definitive" investigations of Brown (1950), Van Bueren (1952) and Wayman et al. (1965) were conducted. They utilized transit circle proper motions either on the system defined by the GC (Van Bueren, 1952) or that of the N30 (Wayman et al., 1965) and more numerous and precise radial velocities from Wilson's catalog. In order to avoid the bias associated with early convergent point solutions, a new technique was introduced in which an approximation to the position of the convergent point was assumed (usually that determined by Smart (1939)), which was then corrected by a least-squares adjustment performed only with the use of the position angles of the observed proper motions. Though this technique proved to be successful in avoiding any obvious bias, information was lost as the magnitudes of the proper motions were removed from the adjustment procedure. Wayman et al. obtained a Hyades distance of 40.6 ± 1.6 parsecs, in good agreement with the value of 40.4 ± 1.1 parsecs determined by Van Bueren. The good agreement between the convergent point solutions led many astronomers to feel that the question of the Hyades distance was closed and that more work would yield only very minor corrections, hardly worth the effort involved. This feeling was shattered by the controversial paper of Hodge and Wallerstein (1966), who argued that the Hyades distance ought to be increased to 48.4 parsecs based on (a) the mean of the parallaxes of Hyades stars in the Yale parallax catalog, (b) the dynamical parallaxes of Hyades visual binaries and (c) the Wilson-Bappu effect.

Modern (1967-present). The current epoch is marked by high precision photographic proper motions and parallaxes obtained through the use of automated measuring machines, precise radial velocities obtained by spectrometers, and the widespread availability of computers. The convergent point solutions have been recast into new forms: the proper motion gradient method employed by Upton (1970), and the radial velocity gradient method recently developed by Griffin et al. (1988). Even though the direct determination of a convergent point is avoided by these techniques, the assumption of a common cluster motion is maintained and the stellar parallaxes are still obtained in a manner similar to the old convergent point solutions. In addition, a new emphasis has developed on cluster distance determined via "astrophysical" or "secondary" means, such as R-I photometry. These techniques produce results which are accorded a good deal of weight and are used as comparisons for the convergent point solutions. Trigonometric parallaxes continue to be left out of the still-primitive adjustment procedures, the mean sometimes being calculated for comparison with the results of an adjustment. Typical modern values for the Hyades' distance are 47.2 ± 1.9 parsecs (convergent point solution, Hanson, 1980), 45.3 ± 1.6 parsecs (mean of trigonometric parallaxes, Upgren, 1974a) and 44.0 ± 0.4 parsecs (R-I photometry, Upgren, 1974b).

Convergent Point Determinations

Derivation of the Classical Convergent Point Condition Equations

Let $\mathbf{x}$ be the location of a star which is a member of the cluster and $\dot{\mathbf{x}}$ be its velocity. Then we may write

$$\begin{aligned} \mathbf{x} &= r\hat{\mathbf{x}}(\alpha, 90^\circ - \delta) = r\begin{pmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{pmatrix} \\ \dot{\mathbf{x}} &= \dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}} \end{aligned} \tag{2-1}$$

where α, δ are the star's right ascension and declination, r is its distance, and $\dot{r}$ its the radial velocity. $\dot{\hat{\mathbf{x}}}$, the time-derivative of the star's direction unit vector, is given by

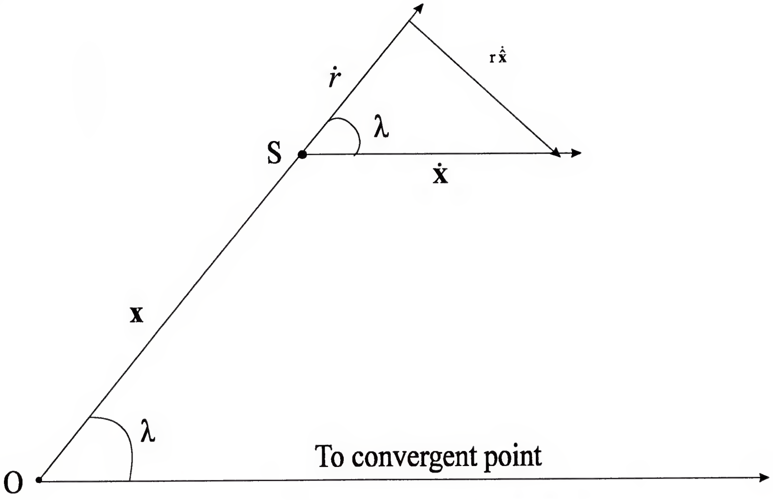

Figure 2-1. Convergent point geometry

$$\dot{\hat{\mathbf{x}}} = \begin{pmatrix} -\dot{\alpha}\cos\delta\sin\alpha - \dot{\delta}\sin\delta\cos\alpha \\ \dot{\alpha}\cos\delta\cos\alpha - \dot{\delta}\sin\delta\sin\alpha \\ \dot{\delta}\cos\delta \end{pmatrix} \tag{2-2}$$

Note that the units of $\dot{\alpha}$ and $\dot{\delta}$ are radians per unit time, whereas the commonly communicated μ_α and μ_δ are in units of " yr^{-1} or " per century. A little algebraic manipulation enables one to show that the magnitude of $\dot{\hat{\mathbf{x}}}$ is the total proper motion of the star, μ:

$$|\dot{\hat{\mathbf{x}}}| = \sqrt{\dot{\alpha}^2\cos^2\delta + \dot{\delta}^2} = \mu \tag{2-3}$$

By definition, $\hat{\mathbf{x}} \cdot \dot{\hat{\mathbf{x}}} \equiv 0$. It can also be shown that

$$\dot{r} = \dot{\mathbf{x}} \cdot \hat{\mathbf{x}}, \tag{2-4}$$

so that the second of equations (2-1) becomes

$$\dot{\mathbf{x}} = (\dot{\mathbf{x}} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}}, \text{ or } r\dot{\hat{\mathbf{x}}} = \dot{\mathbf{x}} - (\dot{\mathbf{x}} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}} \tag{2-5}$$

Now form

$$r\dot{\hat{\mathbf{x}}} \cdot r\dot{\hat{\mathbf{x}}} = (\dot{\mathbf{x}} - (\dot{\mathbf{x}} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}) \cdot (\dot{\mathbf{x}} - (\dot{\mathbf{x}} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}), \tag{2-6}$$

which is equivalent to

$$r^2|\dot{\hat{\mathbf{x}}}|^2 = |\dot{\mathbf{x}}|^2 - \dot{r}^2, \text{ or } r^2\mu^2 = V^2 - \dot{r}^2 \tag{2-7}$$

where V is the magnitude of the star's space velocity (expressed in units of km s^{-1}), i.e., the star's speed. A glance at figure 2-1 gives an alternative expression for the radial velocity:

$$\dot{r} = V\cos\lambda, \tag{2-8}$$

where λ is the angle between the line-of-sight direction and the star's velocity. Substituting the above into equation (2-7) results in

$$r^2\mu^2 = V^2 - V^2\cos^2\lambda = V^2\left(1-\cos^2\lambda\right) = V^2\sin^2\lambda, \text{ or}$$
$$r = \frac{V\sin\lambda}{\kappa\mu} \qquad (2\text{-}9)$$

where κ is the appropriate factor for converting the radial velocities from units of A.U. yr^{-1} to units of km s^{-1} (~ 4.74). In terms of the parallax,

$$\pi = \frac{\kappa\mu}{V\sin\lambda} \qquad (2\text{-}10)$$

Equation (2-10) is a general result, applicable to any star, whether or not it is a cluster member, provided that λ is the angle between the star's position and its velocity.

However, in the case of a group of stars sharing a common velocity, the stars will appear to converge to (or diverge from) a point[3] in the sky, called the convergent (explosion) point. The direction to this point is the same as the group's common space velocity, and may be written

$$\dot{\mathbf{x}} = V\hat{\mathbf{x}}(A, 90^\circ - D) = V\begin{pmatrix} \cos A\cos D \\ \sin A\cos D \\ \sin D \end{pmatrix}, \qquad (2\text{-}11)$$

where A and D are the right ascension and declination, respectively, of the convergent point. Once A and D are known, λ is computed from

$$\cos\lambda = \sin\delta\sin D + \cos\delta\cos D\cos(\alpha - A) \qquad (2\text{-}12)$$

---

[3] In actuality, this will be an area rather than a "point", due to the unavoidable errors in the measured proper motions and the fact that the stars do not, in fact, have exactly the same velocity.

A knowledge of each star's λ, combined with the cluster velocity and the observed proper motions, enable one to carry out the computation of the individual stellar parallaxes by means of equation (2-10). Note that this equation involves only the total proper motion and not its direction, and that it is independent of the radial velocity. It does, however, require knowledge of the cluster velocity. As all stars share this, the velocity may be estimated through a standard least-squares adjustment, with the velocity components as the global variables.

The equations of condition are set up by inverting the second of equations (2-1), which may be rewritten in the form (Eichhorn, 1982):

$$\dot{\mathbf{x}} = r\mathbf{R}_3(-\alpha)\mathbf{R}_2(\delta)\begin{pmatrix} \pi\dot{r} \\ \dot{\alpha}\cos\delta \\ \dot{\delta} \end{pmatrix}, \qquad (2\text{-}13)$$

where $\mathbf{R_2}$ and $\mathbf{R_3}$ are standard rotation matrices, defined by

$$\mathbf{R}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \mathbf{R}_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We solve equation (2-13) with respect to the vector on the right-hand side and obtain

$$\begin{pmatrix} \pi\dot{r} \\ \dot{\alpha}\cos\delta \\ \dot{\delta} \end{pmatrix} = \pi\mathbf{R}_3(-\delta)\mathbf{R}_2(\alpha)\dot{\mathbf{x}}, \qquad (2\text{-}14)$$

which may be explicitly written

$$\begin{pmatrix} \dot{r} - \dot{x}\cos\alpha\,\cos\delta - \dot{y}\sin\alpha\,\cos\delta - \dot{z}\sin\delta \\ \dot{\alpha}\,\cos\delta + \dot{x}\pi\,\sin\alpha - \dot{y}\pi\,\cos\alpha \\ \dot{\delta} + \dot{x}\pi\,\cos\alpha\,\sin\delta + \dot{y}\pi\,\sin\alpha\,\sin\delta - \dot{z}\pi\,\cos\delta \end{pmatrix} = 0, \qquad (2\text{-}15)$$

where $\dot{x}$, $\dot{y}$ and $\dot{z}$ are the cartesian components of $\dot{\mathbf{x}}$ (units are km s^{-1}).

These are the desired condition equations necessary for the determination of the cluster velocity. Note that they are nonlinear if one regards the parallax, π, as an observation, because of the terms $\dot{x}\pi$, $\dot{y}\pi$, and $\dot{z}\pi$. Past investigators have employed various techniques for circumventing the failure of equations (2-15) to be linear, which shall be considered in some detail in the following sections.

Charlier's Method

Before 1945, investigators employed a method developed by Charlier (1916) to obtain the distances to the stars of the Hyades cluster. This technique ignored the first of equations (2-15) altogether, wasting practically all the information contained therein. The parallax was then eliminated from the second and third of equations (2-15), resulting in the condition equation

$$a\dot{x} + b\dot{y} - c\dot{z} = 0, \qquad (2\text{-}16)$$

where

$$\begin{aligned} a &= \dot{\alpha}\cos\delta\cos\alpha\sin\delta - \dot{\delta}\sin\alpha \\ b &= \dot{\alpha}\cos\delta\sin\alpha\sin\delta + \dot{\delta}\cos\alpha \\ c &= \dot{\alpha}\cos^2\delta \end{aligned}$$

In order to simplify the computations, equation (2-16) was rewritten as

$$a\frac{\dot{x}}{\dot{y}} - c\frac{\dot{z}}{\dot{y}} + b = 0 \qquad (2\text{-}17)$$

and a regression performed to estimate values for the ratios of the velocity components. The coordinates of the convergent point were then obtained from equations (2-11). Note that the quantity minimized in this "adjustment" has nothing to do with the sums of the squares of any residuals. The only justification for employing this wretched model is that it leads to a result (after a fashion), with the minimum arithmetic effort, an important consideration in those days! V is, of course, still missing completely; but equation (2-8), recast in the form

$$V = \dot{r} \sec \lambda, \tag{2-18}$$

became the condition equation for the second of the adjustments, now with the aim to estimate V, involving the observed radial velocities. The cluster velocity having thus been determined by these two separate adjustments, the individual stellar parallaxes were computed by means of equation (2-10).

At this point, one may ask why Smart and other early investigators went to the trouble of eliminating the parallax from equation (2-15), when they could obtain the velocity components directly from considering only the first of these equations, i.e., by working with the radial velocities exclusively. The answer is two-fold: one, the Hyades cluster is fairly concentrated, with most stars lying within a few degrees of the cluster's center, and two, the precision of the available radial velocities was rather poor (standard errors greater than 5 km s^{-1}). Even with today's high precision radial velocities (standard errors of less than 1 km s^{-1}), these reasons still make the first of equations (2-15) a poor choice for an adjustment condition equation. Why a simultaneous adjustment of equations (2-15) and (2-16) was not performed remains, however, still puzzling. Surely these investigators were familiar with the concept of weight!

Many investigators used this technique to estimate distances to the Hyades stars, including Plummer (1912), Rasmuson (1921), Smart (1939), Losert (1939), and Pearce (1955). All, with the exception of Boss (1908), obtained a Hyades distance under 40 parsecs, with Smart's value of 36.3 ± 1.2 parsecs being representative. Table 2-1 summarizes the results of some of these investigations.

Table 2-1. Early Hyades distance determinations.

Investigator	**α CP**	**δ CP**	**Distance (pc)**	**No. stars**
Boss (1908)	92°.82 ± 2°.2	6°.92 ± 0°.44	41.7	41
Losert (1939)	92°.75 ± 1°.6	6°.87 ± 1°.6	37	10
Smart (1939)	91°.83 ± 0°.66	8°.07 ± 0°.26	36.3 ± 1.2	72
Pearce (1955)	91°.85 ± 0°.65	9°.1 ± 0°.25	37 ± 1	74

Seares' Criticism of Charlier's Method

So matters stood until 1945, when Frederick Seares published a critique of Charlier's method, in which he showed that this technique would introduce a bias, i.e., systematically underestimate the distances to Hyades members. The problem arises because the coefficients of condition equations (2-17) contain the observed and therefore unavoidably error-affected proper motion components of the star. Incorporating these errors into the condition equations yields

$$a'\frac{\dot{x}}{\dot{y}} - c'\frac{\dot{z}}{\dot{y}} + b' = 0 \qquad (2\text{-}20)$$

where a', b', and c' are given by

$$\begin{aligned} a' &= a + \varepsilon_{\mu\alpha}\cos\delta\cos\alpha\sin\delta - \varepsilon_{\mu\delta}\sin\alpha \\ b' &= b + \varepsilon_{\mu\alpha}\cos\delta\sin\alpha\sin\delta + \varepsilon_{\mu\delta}\cos\alpha \\ c' &= c + \varepsilon_{\mu\alpha}\cos\delta \end{aligned} \qquad (2\text{-}21)$$

with $\varepsilon_{\mu\alpha}$ and $\varepsilon_{\mu\delta}$ being the errors in the proper motion components. The primed coefficients are the ones unwittingly used in Charlier's method; the unprimed coefficients are the "true" values. The ordinary least-squares solution of equations (2-20) is given by

$$\Delta = -\left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{F}_{0} \qquad (2\text{-}22)$$

where Δ is the vector of parameter estimates, and

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} a_1' \cdots a_m' \\ -c_1' \cdots -c_m' \end{pmatrix},\ \mathbf{F}_0 = \begin{pmatrix} -b_1' \\ \vdots \\ -b_m \end{pmatrix}, \tag{2-23}$$

with m being the number of stars. $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ has form

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} \sum_{i=1}^{m} a_i' a_i' & \sum_{i=1}^{m} -a_i' c_i' \\ \sum_{i=1}^{m} -a_i' c_i' & \sum_{i=1}^{m} c_i' c_i' \end{pmatrix} \text{ or, expanding the first column,}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} \sum_{i=1}^{m}\left(aa + 2a\varepsilon_{\mu\alpha}\Lambda_1 - 2a\varepsilon_{\mu\delta}\Lambda_2 - 2\varepsilon_{\mu\alpha}\varepsilon_{\mu\delta}\Lambda_1\Lambda_2 + \varepsilon_{\mu\alpha}{}^2\Lambda_1^2 + \varepsilon_{\mu\delta}{}^2\Lambda_2^2\right) & \sum_{i=1}^{m} -a_i' c_i' \\ -\sum_{i=1}^{m}\left(ac + c\varepsilon_{\mu\alpha}\Lambda_1 - c\varepsilon_{\mu\delta}\Lambda_2 + a\varepsilon_{\mu\alpha}\Lambda_3 - \varepsilon_{\mu\alpha}\varepsilon_{\mu\delta}\Lambda_2\Lambda_3 + \varepsilon_{\mu\alpha}{}^2\Lambda_1\right) & \sum_{i=1}^{m} c_i' c_i' \end{pmatrix}$$

where

$$\begin{aligned} \Lambda_1 &= \cos\delta\cos\alpha\sin\delta \\ \Lambda_2 &= \sin\alpha \\ \Lambda_3 &= \cos\delta \end{aligned}$$

Note that the elements of $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ differ from the "true" elements by terms involving the errors in the proper motion components. The terms containing $\varepsilon_{\mu\alpha}{}^2$ and $\varepsilon_{\mu\delta}{}^2$ account for most of the differences, as the linear and mixed terms mostly cancel. These terms will introduce systematic errors into the parameter estimates, and hence the direction to the convergent point.

In applying the above to Smart's data, Seares revised the equatorial coordinates of the Hyades convergent point from Smart's original (91°.83, 8°.07) to (93°.48, 7°.53). The alteration in the convergent point direction resulted in a change to the cluster distance by some 7%, moving the center of the Hyades out by approximately 3 parsecs to 39.2 ± 1.2 parsecs. It should perhaps be emphasized that Seares kept only the terms involving $\varepsilon_{\mu\alpha}{}^2$

and $\varepsilon_{\mu\delta}^2$ in the normal equations (apparently because he trusted the linear — $\varepsilon_{\mu\alpha}$ and $\varepsilon_{\mu\delta}$ — and mixed — $\varepsilon_{\mu\alpha}\varepsilon_{\mu\delta}$ — terms to cancel), rendering his correction to the Hyades distance no more than a first order approximation. Nonetheless, his paper caused a minor stir in the astrometric community which resulted in the development of a new convergent point technique.

The Classical Convergent Point

Brown (1950) also pointed out the shortcomings in Charlier's method, but was equally critical of Seares' regression correction. His dissatisfaction with the techniques then employed caused him to evaluate different approaches in solving for the convergent point, leading to his conclusion that it would be better to use the position angles of the individual stellar proper motions rather than the proper motion components. While keeping the assumption of identical stellar velocities, Brown proposed the abandonment of obtaining the condition equations from equations (2-15) in favor of a derivation from the equation giving the position angle at the star's position (α, δ) of the great circle through (α, δ) and the coordinates of the convergent point (A, D). He also advocated obtaining a solution via differential corrections or an alternate maximum likelihood technique rather than using a least squares regression. However, his application of the maximum likelihood approach to the Ursa Major association met with very limited success, as the numerical computations were quite complicated for the available computational facilities, i.e., mechanical calculators.

Later investigators (van Bueren, 1952; Wayman et al., 1965) adopted Brown's concept of using the proper motion position angles to determine the convergent point, but returned to least squares as the technique used to obtain the solutions. Starting with the great circle equation used by Brown,

$$\cot\theta = \tan D \cos\delta \csc(A-\alpha) - \sin\delta \cot(A-\alpha), \tag{2-24}$$

with θ being the position angle at the star's position (α, δ) of the great circle through (α, δ) and the coordinates of the convergent point (A, D), they performed a differentiation with respect to A and D. This resulted in the condition equation for the adjustment,

$$B\Delta A - C\Delta D = \theta_0 - \theta \ , \tag{2-25}$$

where θ_0 are the stellar proper motion position angles (measured from the direction to the pole of rotation), determined from

$$\tan\theta_0 = \frac{\dot{\alpha}\cos\delta}{\dot{\delta}}, \tag{2-26}$$

and B and C are given by

$$\begin{aligned} B &= \cot(A_0 - \alpha)\cos\theta\sin\theta - \sin\delta\sin^2\theta \\ C &= \sec^2 D_0 \cos\delta\csc(A_0 - \alpha)\sin^2\theta \end{aligned} \tag{2-27}$$

Note that B and C do not depend on any observed quantities (the star positions are known with great precision and so are treated as constants), thereby avoiding the regression error associated with Charlier's technique. The parameters obtained by the adjustment are ΔA and ΔD, these being the corrections to the assumed position of the convergent point (A_0, D_0). The coordinates of the convergent point were then found from the simple relations

$$\begin{aligned} A &= A_0 + \Delta A \\ D &= D_0 + \Delta D \end{aligned} \tag{2-28}$$

Once these were obtained, the cluster velocity and the individual stellar parallaxes were determined as in Charlier's method. Brown, in conformance with then and — alas — even now prevailing common practice makes no attempt to justify the minimization of the sums of the squares of the position angle residuals, although such a justification could be contrived by invoking the central limit theorem.

The first, and perhaps best known, investigation using the above technique was that of van Bueren (1952), who placed the Hyades convergent point at (94°.62 ± 0°.53, 7°.48 ± 0°.18) and the distance to the cluster center at 40.4 ± 1.1 parsecs. Since then there have been many investigations of the Hyades distance using the classical convergent point approach, including a detailed study by Wayman et al. (1965). The results of some of these investigations are summarized in table 2-2 and the positions of the convergent points are displayed in figure 2-2. Note the gradual progression of the convergent point towards higher right ascensions and smaller declinations, due in part to the systematic differences among the proper motion sets used in the solutions. These differences will be discussed in some detail in a later section.

Table 2-2. Classical Hyades convergent point solutions.

Investigator	**α CP**	**δ CP**	**Distance (pc)**
van Bueren (1952)	94°.62 ± 0°.53	7°.48 ± 0°.18	40.4 ± 1.1
Heckman and Lübeck (1956)	94°.0	7°.6	41.2
Wayman et al. (1965)	94°.13 ± 0°.49	7°.68 ± 0°.19	40.6 ± 1.6*
Corbin et al. (1975)	96°.3	7°.27	43.55 ± 0.74
Hanson (1975)	102°.5 ± 3°.7	5°.9 ± 0°.8	48.4 ± 4.8
Morris and Luyten (1983)	95°.2 ± 0°.5	7°.8 ± 0°.2	42.7
Schwan (1991)	97°.68 ± 0°.25	5°.98 ± 0°.18	47.9 ± 0.8

Gradient Techniques

Upton (1970) developed a technique in which the gradients in either the right ascension proper motions or the declination proper motions could be combined with the average radial velocity of the Hyades stars to obtain the cluster's distance. Although the method does not directly determine the position of the convergent point, it is a "convergent point" technique in the sense that it retains the assumption of a common space velocity for the Hyades stars. Hanson notes that ". . . the method of analysis is simply the CP principle treated in different mathematical form" (Hanson, 1975, p. 395)[4] , pointing out that Upton's equations can be derived by differentiating equation (2-9).

The basic concepts behind Upton's approach can be grasped by considering the Hyades to be spherical in shape, having some finite diameter, s, and center at some distance, r. Then simple trigonometry leads to the relation

$$\theta = \frac{s}{r}, \tag{2-29}$$

where θ represents the angular diameter of the cluster. Under the assumptions that the cluster is of small angular extent and undergoes no expansion or contraction, the differentiation of this with respect to time results in

$$\begin{gathered} \dot{\theta} = -\frac{s}{r^2}\dot{r}, \text{ or} \\ \frac{\dot{\theta}}{\theta} = -\frac{\dot{r}}{r} \end{gathered} \tag{2-30}$$

Simply put, this equation states that to first order, the distance to the Hyades can be determined by dividing its mean radial velocity by the fractional decrease in the cluster angular diameter. This decrease in the cluster's angular size as it recedes can be determined from the convergence of the member stars' proper motions; indeed, Upton (1970) shows that equation (2-30) can be approximated by

$$\frac{\dot{r}}{r} = -\frac{d\mu_\alpha}{d\alpha} = -\frac{d\dot{\delta}}{d\delta} \tag{2-31}$$

---

[4] This statement seems somewhat contrived. The simple convergent point method is based on the first time derivatives of the location components in polar coordinates, while the gradient techniques regard the (much less precise!) second time derivatives of these coordinates as the observed quantities, thus neutralizing constant biases. Insofar as Hanson implies that both classes of methods, applied to the same observations, will yield the same results, he is wrong.

where the terms on the right represent the proper motion gradients in right ascension and declination, respectively. It should be noted that the cluster distance can be obtained from either gradient, and much has been made of the notion that any differences between the distance determined from the right ascension proper motion components and that determined from the declination components should give some idea of the accidental and systematic errors in these proper motions (see Upton, 1970 and Hanson, 1975). While this may be true of the accidental errors involved, it seems not to be the case with the systematic errors, as Upton's Hyades distance determination fails to show the systematic errors in the transit circle proper motions used in his adjustment. Similarly, Hanson's (1975) application of the technique failed to reveal the magnitude dependency in his photographic proper motion estimates (McAlister, 1977).

A variant of Upton's approach, one using the radial velocity gradient, was developed by Gunn et al. (1988). This technique obtains its basic equation by differentiating equation (2-8) with respect to λ:

$$\frac{d\dot{r}}{d\lambda} = V \sin\lambda = \dot{r} \tan\lambda \tag{2-32}$$

Substitution of the above into equation (2-9) gives the result

$$r = \frac{\left(\dfrac{d\dot{r}}{d\lambda}\right)}{\kappa\mu}, \tag{2-33}$$

where μ is the mean cluster proper motion. The accurate determination of the radial velocity gradient has only recently been made possible by high precision radial velocities measured by Detweiler (Detweiler et al., 1984) and Griffin (Griffin et al., 1988), as the scatter in the Wilson radial velocities was much too large (of the order of 5 km s^{-1}).

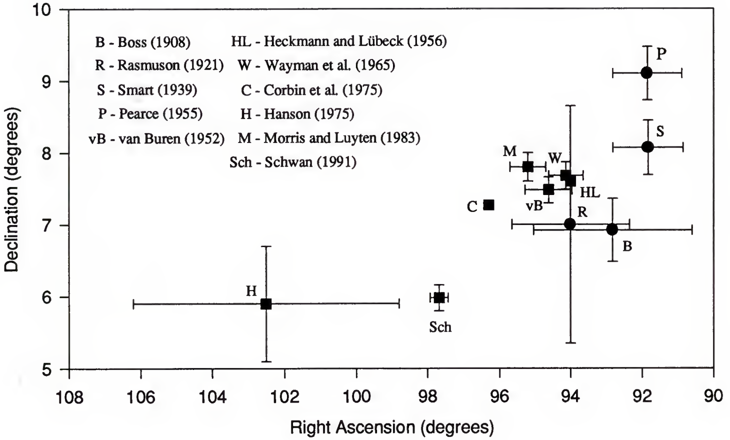

Figure 2-2. Hyades convergent point positions (orientation 1950.0)

Table 2-3 summarizes the Hyades distance determinations obtained using the gradient techniques outlined above. There is the same large scatter in the Hyades distance as is seen in the classical convergent point investigations, illustrating the difficulties involved in obtaining absolute proper motions and radial velocities free from systematic effects.

Table 2-3. Gradient technique distance determinations.

Investigator	Gradient Type	Distance (pc)
Upton (1970)	Proper Motions	41.6 ± 1.0
Hanson (1975)	Proper Motions	48.4 ± 4.4
Gunn et al. (1988)	Radial Velocity	45.4 ± 2.1

Secondary Distance Determinations

The Hodge and Wallerstein Controversy

In the mid-1960's, Hodge and Wallerstein (1966) created a stir in astronomical ranks by suggesting that the distances to the Hyades as derived from the convergent point determinations were too small by some 8 parsecs. They based their argument on the results of investigations involving so-called "secondary" indicators — trigonometric parallaxes, dynamical parallaxes of Hyades visual binaries and the widths of the K-emission line cores in four Hyades yellow giants (the Wilson-Bappu effect). The distances obtained from these methods and presented by Hodge and Wallerstein in their paper are summarized in table 2-4.

Although there was much initial resistance, it has now become accepted that the "secondary" Hyades distance indicators should be given weight equal to convergent point techniques, especially with the evidence mounting that proper motions are quite sensitive to various systematic errors. As Hodge and Wallerstein (1966) point out, a systematic error of only 0.0008 " yr^{-1} deg^{-1} in the declination proper motions can cause the Hyades distance to change by almost 20%. A glance at the previous section also shows that convergent point solutions incorporating modern proper motions placed on the FK5 system or tied to a reference frame of external galaxies (which are presumably less affected by systematic errors) yield distances in rough agreement with Hodge and Wallerstein's value. Improved instrumentation and more precise observations have led to the use of many "secondary" indicators in determining the Hyades distance, and some of these are explained briefly in the following paragraphs.

Table 2-4. Hodge and Wallerstein Hyades distances.

Indicator	Distance (pc)
Trigonometric parallaxes	48.5 ± 7.8
Dynamical parallaxes of visual binaries	44.4
Widths of K-emission lines	52.1
Average of the above	48.3

Trigonometric Parallaxes

The simplest of the secondary indicators is also the most direct, as the measurement of a star's trigonometric parallax determines its distance: the inverse of that parallax. Until the past two decades or so, parallaxes were of fairly low precision, with standard errors of 0".02 or more. As the distance to the center of the Hyades lies somewhere between 45 and 50 parsecs (parallax between 0".020 and 0".022), parallaxes of Hyades stars had an error roughly equivalent to the parallax itself, sometimes resulting in the determination of physically unrealistic negative or very small parallaxes. However, the advent of automated measuring machines, the MAP (used at the Allegheny Observatory, see Gatewood et al., 1992), CCD cameras and finally, the astrometric satellite HIPPARCOS have improved the precision of modern parallax determinations to about 0".001, a gain of an order of magnitude over the past. This improved precision has made possible fairly reliable estimates of the distance to the cluster center from the parallaxes alone, provided the number of stars considered in the estimate is more than a handful. Table 2-5 summarizes some of the Hyades distance estimates obtained by averaging the measured Hyad parallaxes.

Table 2-5. Estimates of Hyades distance incorporating parallaxes.

Investigator	Distance (pc)	No. of stars
Upgren (1974)	45.7 ± 3.8	11
Klemola et al. (1975)	43.5 ± 2.8	18
Upgren et al. (1990)	45.7 ± 2.1	23
Patterson and Ianna (1992)	45.9 ± 2.1	10

Photometric Approaches

Another class of techniques that has enjoyed a fair amount of popularity in recent Hyades distance determinations incorporates photoelectric photometry of Hyades members, usually intermediate band (Mannery and Wallerstein, 1971) or (R-I) (Upgren, 1974; Eggen, 1982). Much has been made of the (R-I) determinations, as the (R-I) index is not strongly dependent on metal abundance. The procedure followed by Upgren (1974b) was to use trigonometric parallaxes (after correction for known systematic effects) of non-member dwarfs in the Hyades field to establish a (M_V, R-I) main sequence for these background stars, which have (R-I) values between 0.35 and 1.05. A linear fit was then made to the photometry of the Hyades members to establish a (V, R-I) sequence. A comparison of this sequence to the (M_V, R-I) sequence for the background stars yields

Table 2-6. Photometric distance determinations.

Investigator	Colors used	Distance (pc)
Eggen (1969)	(M_I, R-I)	40.5 ± 1.9
Mannery and Wallerstein (1971)	(V, V-r)	44.5 ± 2.0
Upgren (1974b)	(M_V, R-I)	44.0 ± 0.8
Eggen (1982)	(M_I, R-I)	43.6 ± 2.0
Cameron (1985)	(V, B-V)	44.9 ± 2.3

the Hyades distance modulus, m-M.[5] Eggen (1969, 1982) followed a somewhat similar procedure, save that he used (I-M_I) as his distance modulus, where M_I is the absolute I magnitude of the Hyades stars. Both investigators obtained a Hyades distance in the neighborhood of 44 parsecs, although the earlier determination by Eggen gives a distance in agreement with the ~41 parsec distance of the classical convergent point solutions. There are no obvious reasons for this discrepancy between Eggen's determinations, a point made by Mannery and Wallerstein (1971), who obtained a Hyades distance of 44.5 parsecs based on their BVr photometry.

Cameron (1985) came up with a technique that used UBV photometry of Hyades and field stars to obtain the cluster distance modulus. After selecting a parallax-limited set of field stars redder than the Sun ($0.68 < (B-V) < 1.00$) from Gliese's catalogue, he applied the Lutz-Kelker corrections and corrections for metallicity, thereby obtaining a zero age main sequence (ZAMS). This was compared to a Hyades main sequence established by Hyades dwarfs with $0.1 < (B-V) < 1.1$, resulting in a Hyades distance modulus of 3.26 ± 0.11, corresponding to a distance of 44.9 ± 2.3 parsecs.

The Wilson-Bappu Effect

One of the arguments Hodge and Wallerstein (1966) presented against the Hyades distances determined from the classical convergent point investigations involved the Wilson-Bappu effect. In brief, the Wilson-Bappu effect states that there is a linear relationship between the logarithm of the CaII H and K emission-line core width and the absolute visual magnitude of the star. Hodge and Wallerstein (1966) performed a re-determination of the parameters of this linear relationship using seven background stars with known parallaxes and γ Andromedae A. Fitting the Hyades stars to this line resulted in a distance of 52 parsecs to the cluster center. However, Wilson (1967) challenged this value based on his recalibration of the Wilson-Bappu effect, involving 65 stars with "good" parallaxes, finding no substantial disagreement with a Hyades distance of 41 parsecs. Lutz (1970) settled the debate by showing that Wilson's linear regression introduced systematic errors into the parameters by allowing errors only in the emission-line widths and ignoring those in M_V. The application of an improved adjustment technique allowing errors in both observed quantities enabled Lutz to derive a Hyades distance of 44.3 parsecs, in good agreement with other distance indicators. See table 2-7 for a summary of these determinations.

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[5] The distance is obtained from the distance modulus by noting that $m - M + 5 = 5 \log d$, where d is the distance in parsecs.

Table 2-7. Hyades distance estimates obtained from the Wilson-Bappu effect.

Investigator	**Distance (pc)**
Hodge and Wallerstein (1966)	52
Wilson (1967)	41
Lutz (1970)	44.3

Techniques Involving Binaries

Using orbit solutions for several Hyades binaries and applying the local mass-luminosity relation, Hodge and Wallerstein (1966) derived a Hyades distance of 44.4 parsecs. McClure (1982) followed a similar approach, but he was able to directly determine the mass of the Hyad vB 22 (HD 27130), a double-lined spectroscopic and partially eclipsing binary. After assuming a slope for the mass-luminosity relation, he was able to derive a cluster distance of 49.4 ± 1.1 parsecs from the available data on 10 visual binaries belonging to the Hyades. Peterson and Solensky (1987) revised this distance down to 47.0 ± 1.2 parsecs, based on a new estimate of the period of one of the binaries, Finsen 342, and a re-evaluation of the available data on vB 22. Table 2-8 summarizes these investigations.

Table 2-8. Hyades distance determinations using binary star data.

Investigator	Distance (pc)
Hodge and Wallerstein (1966)	44.4
McClure (1982)	49.4 ± 1.1
Peterson and Solensky (1987)	47.0 ± 1.2

Constraints Imposed by Stellar Models

With the increase in computing power over the past few decades, it has become possible to construct elaborate stellar models and produce theoretical color-magnitude diagrams and a mass-luminosity relation for the Hyades, which can then be compared to the observations. Problems became evident three decades ago when Iben (1963) and Bodenheimer (1965) were unable to produce models that matched the observed Hyades color-magnitude diagram and the mass-luminosity relation as derived from distances based on convergent point solutions. Iben could achieve consistency only by assuming $Y=0.26$, $Z=0.0625$ in his model calculations, values ruled out by spectroscopic and photometric observations of Hyades members. The high metal abundance was felt to be necessary to explain the differences between the observed Hyades mass-luminosity relation and that of the solar neighborhood. According to Eggen (1963), a Hyad of 0.65 solar masses would have a luminosity equal to that of the Sun, making the Hyades stars overluminous for their masses. Faulkner (1967) used his quasi-homology relations to point out that an increase in the Hyades distance from 40.4 parsecs to 44.4 parsecs could resolve the differences between the models and observations without having to postulate large metal abundances. The same distance was obtained by Wallerstein and Hodge (1967), based on models produced by Demarque (1967). Later on, Hardorp (1980) used his interior and atmosphere calculations to obtain a cluster distance of 45 parsecs, and recently VandenBerg and Bridges (1984) came up with a value of 49.0 ± 1.1 parsecs. It should be noted that as the estimated Hyades distance increased, the cluster mass-luminosity relation moved towards agreement with that of the solar neighborhood. Table 2-9 is a short summary of Hyades distance estimates based on stellar models.

Table 2-9. Hyades distance constraints based on stellar models.

Investigator	Distance (pc)
Faulkner (1967)	> 44.4
Wallerstein and Hodge (1967)	> 44.4
Hardorp (1980)	45
VandenBerg and Bridges (1984)	49.0 ± 1.1

Critique of Past Hyades Distance Determinations

As a glance at figure 2-3 will readily show, the distance to the Hyades is still very much an open question, the general trend being for the cluster distance to increase with time. Indeed, if we were to accept the error bars in figure 2-3 at face value, it would seem that the Hyades cluster is receding from the solar system at a speed of 0.66c, despite having a measured radial velocity of ~39 km s^{-1}! This is, of course, nonsense, but it serves to illustrate the point that the cluster distance determinations are not as accurate as the given errors would indicate.

As the author's training lies in the field of astrometry, it is somewhat difficult and perhaps inappropriate to be very critical of the Hyades distance estimates achieved through the use of "secondary" astrophysical indicators. However, the fact that many of these indicators rely on trigonometric parallaxes in the calibrations is something that cannot be ignored in the evaluation of the precision or accuracy of their distance estimates. The best currently available parallaxes have standard errors of 0".001, which is more than 5% of a typical Hyad parallax, while the older parallaxes used in many calibrations, such as in that for the photometric determinations, have quoted errors ten times this amount. Also, it would seem that the naively erroneous application of linear regression is not restricted to just the early convergent point investigators. As Lutz (1970) showed, Wilson

(1967) commited a similar mistake in his calibration of the Wilson-Bappu effect, thereby underestimating the distance to the Hyades by slightly over three parsecs. Finally, it should be pointed out that some of these indicators, such as early stellar models, use observations of Hyads as a theoretical check, making their use as estimators of the Hyades distance seem somewhat like circular arguments.

Given their quoted lesser precision, it is somewhat ironic that the distance estimates derived from the secondary indicators should nowadays be given as much weight as the estimates obtained from the more precise, "fundamental" convergent point solutions. To some extent, this has not been due to any deficiencies in the convergent point techniques, but is a consequence of the acknowledgment that the various sets of proper motion and radial velocity data suffer from systematic errors that produce the wide scatter observed in the Hyades distance estimates. Nonetheless, the convergent point techniques do suffer from some serious deficiencies, which are summarized in the paragraphs below.

All stars are forced to have the same velocity. Even though the velocity dispersion of the stars which are members of the Hyades is small (~0.25 km s^{-1}; see van Bueren, 1952, and Gunn et al., 1988), the precision of modern radial velocities is approaching this value (Griffin et al., 1988) and shall soon reach a point where the internal motions of the cluster members will no longer be "lost in the noise". As the observations become precise enough to discern the individual stellar motions, so too must the data reduction techniques allow each star to "go its own way".

Trigonometric parallaxes are ignored. Despite the gains in precision, trigonometric parallaxes are left out of convergent point solutions entirely and are relegated to the role of a "secondary" distance estimator. This should not be the case, as parallaxes are fundamental observations, with just as much importance as proper motions and radial velocities. It should be noted that equations (2-15) explicitly contain the parallax,[6] making its incorporation into an adjustment procedure simple for someone familiar with the appropriate techniques. All that needs to be done is to use all three of equations (2-15), rather than follow the past procedure of ignoring the first and eliminating the parallax from the second and third equations. This way no fundamental information is ignored, and, as a significant number of Hyads (> 40) have observed parallaxes, such an adjustment is bound to yield a more accurate cluster distance estimate.

Separate adjustments are made for each data type. In the convergent point solutions, the proper motions are used to obtain an estimate of the convergent point location, and then a separate adjustment involving the radial velocities is performed to obtain the cluster space speed. This approach is outdated; modern adjustment techniques and computing power allow for an adjustment of all fundamental data simultaneously, thereby achieving a more accurate estimate. The same criticism may be leveled at the proper motion gradient methods, which incorporate only one proper motion component into each adjustment.

The data are not used in such a manner as to extract the most information. One of the most serious criticisms that can be made of the classical convergent point solutions is that they ignore information by considering only the position angles of the proper motions in the estimation of the position of the convergent point. The magnitudes of the proper motions also carry information, and discarding them can only result in a less precise estimate, even though the precision may seem better. This mistake is often compounded by the mysterious and arcane ways in which stars are selected for incorporation into the adjustment. For example, Schwan (1991) presents 145 stars as members of the cluster, yet uses only 62 of these to estimate the location of the convergent point.

In light of better observations and ever-increasing computer power, it would seem time to retire the classical techniques in favor of some other procedure that can incorporate all fundamental data and relax the physically absurd requirement that all stars must have the same velocity. However, before these techniques can be retired, a better procedure must be found to take their place. The purpose of this work has been the development of just such a procedure; its outline and application are presented in the pages that follow.

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[6] Actually, the distance.

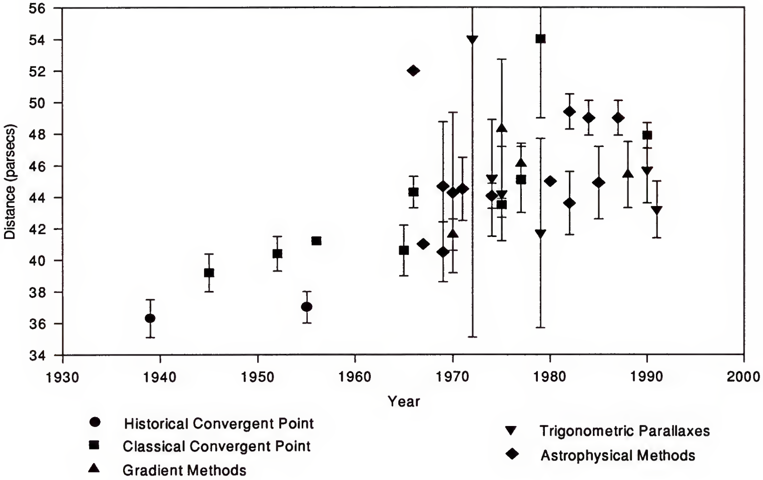

Figure 2-3. Hyades distance determinations

CHAPTER 3
LEAST SQUARES WITH PROBABILISTIC CONSTRAINTS

Introduction

The traditional approach in solving the overdetermined[1] system **Ma** = **x**, where **a** is the vector of parameters (unknowns) and **x** is the vector of observable quantities (observations), is through the application of the principle of least squares. In this technique, we seek estimates of **a** such that

$$\sum_{i=1}^{m}\left[\sum_{j=1}^{n}\left(M_{i,j}a_j\right)-x_i\right]^2=\text{minimum,} \tag{3-1}$$

where m is the number of observations and n is the number of parameters. The fact that the magnitude of the innermost sum is squared gives the method its sobriquet of "least squares". Other techniques do exist, in particular the method of minimum sum, which obtains the parameters by setting

$$\sum_{i=1}^{m}\left|\sum_{j=1}^{n}\left(M_{i,j}a_j\right)-x_i\right|=\text{minimum.} \tag{3-2}$$

While the method of minimum sum is more robust[2] than the method of least squares, making it useful for identifying outliers and observational "blunders", it is also true that the technique does not give the most accurate estimates of **a** if **x** is free from bias[3]. Furthermore, the minimum sum does not yield unambiguous estimates of the parameters (Eichhorn,1993). Nonetheless, Branham (1990), who has developed an algorithm for its implementation, suggests this technique because of its usefulness in preprocessing.

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[1] An overdetermined system is a system of independent equations in which the number of equations exceeds the number of unknowns.

[2] By "robust", it is meant that the parameter estimates obtained by the technique are not strongly dependent on the distribution function of the observations.

In the flurry to develop more robust adjustment techniques, the method of least squares is occasionally criticized as being too specialized, especially with regard to the distribution function of the (unknown) errors. Many adjustors of data labor under the mistaken impression that the least-squares technique requires the errors in the observations to obey the so-called normal law,

$$\varphi(\xi) = Ke^{\frac{-1}{2\sigma^2}\xi^T\xi}, \tag{3-3}$$

where σ is the standard deviation of the observations, and ξ, the vector of (unknowable) observation errors, is given by $\xi = \mathbf{x} - \mathbf{x}_0$, $\mathbf{x}_0$ being the vector of bias-free observations. This is not so, as the principle of maximum likelihood can be invoked to show that the most accurate parameter estimates are obtained by a least-squares adjustment if the distribution of errors is described by any function which shares its (unique) maximum with that of the Gaussian error function, not just that given by equation (3-3). It should also be noted that the Gauss-Markov theorem, which states that the least-squares estimator has the minimum variance of the class of linear unbiased estimators, places no restriction on the form of the distribution function of the errors.

In any adjustment, whether by least squares or some other technique, there must exist an adjustment model, that is, a set of equations relating the observations to the unknown parameters. Often this model is a simplified approximation of the actual relationship, which may be unknown, unknowable or deemed to be too complicated to be considered in the adjustment. Whatever the reason(s), any deficiencies in the model will result in "systematic" errors in the parameter estimates, making it important that the adjustment model conform as closely as possible to reality. Many of the "systematic" differences between observations obtained from different observatories or by different instruments can be explained by use of inadequate adjustment model, though it may be impossible in many instances to determine models sufficiently close to reality such that the systematic effects disappear or become unnoticeable.

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[3] Bias can be defined mathematically by $\varphi(\xi) = Ke^{\frac{-1}{2\sigma^2}(\xi-\alpha)^T(\xi-\alpha)}$, where α is the bias. For example, the mean, $\overline{x}$, of a set of measurements of the width of a box may be thought of as an unbiased estimator of this width, whereas $\overline{x} - 5$ is a biased estimator of the box's width.

General Least Squares

Introduction

In general least squares, the restrictions imposed in the ordinary least squares (OLS) technique used by most scientists and engineers[4] are abandoned. In the OLS techniques it is assumed that

1) the parameters enter linearly into the equations of condition,
2) each condition equation involves only one observation,
3) there exist no constraints among the parameters, and
4) the observations have the same precision and are uncorrelated.

Generalized least-squares techniques, such as those outlined by Jefferys (1980) and Eichhorn (1993) remove the above limitations by allowing

1) the equations of condition to be nonlinear with regard to the parameters,

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[4] It should be pointed out that many scientists and engineers are unaware of the restricted circumstances required for an OLS adjustment, sometimes resulting in the misapplication of the technique and consequently, biased parameter estimates.

2) each equation of condition to contain more than one observable,

3) constraints (even probabilistic ones) to exist among the parameters, and

4) the observables to be correlated and of varying precision .

Note that the relaxation of restriction 4) requires that the covariance matrix, **σ**, of the observations must be known, at least to within a constant factor, before the adjustment is attempted. This underscores the importance of publishing the errors in the observations along with the observations, as the variances computed from these errors comprise the elements of **σ**. Without these errors, it becomes practically impossible to achieve an accurate parameter estimation at first attempt. One can, of course, always estimate variances and where appropriate, covariances by analyzing the residuals from a preliminary adjustment.

Derivation of the Normal Equations

Let $\mathbf{x_0}$ be the vector of unbiased observations, with associated covariance matrix **σ**. The observations are estimates of the actual, "true" values, **x**, and the two are related through $\mathbf{x} = \mathbf{x_0} + \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the vector of corrections to the observations. We seek to solve the set of model equations, more commonly known as the equations of condition,

$$\begin{aligned} \mathbf{F}(\mathbf{x_0} + \boldsymbol{\xi}, \mathbf{a}) &= \mathbf{0} \\ \mathbf{G}(\mathbf{a}) &= \mathbf{0}, \end{aligned} \tag{3-4}$$

for the vector **a** of parameters. **G(a)** is a vector of functions, which when set to 0, constrain the adjustment parameters; they are neither implicit nor explicit functions of any observable quantities and establish the relationships (if any) among the parameters, **a**. Equations (3-4) must be solved for the **a** while simultaneously minimizing

$$\mathrm{S} = \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{\sigma}^{-1} \boldsymbol{\xi} \tag{3-5}$$

The solution is best obtained through the application of Lagrange multipliers. Realizing that the product of **F** and **G** with any number is 0 at the solution, equation (3-5) has as a consequence, that also

$$S = \xi^T\sigma^{-1}\xi - 2\mathbf{F}^T\Lambda - 2\mathbf{G}^T\Xi, \tag{3-6}$$

at the solution (i.e., when $\mathbf{F} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$), where Λ, Ξ are yet undetermined vectors of Lagrange multipliers. The minimum of S is found by setting to 0 the partial derivatives of equation (3-6) with respect to the observations and the parameters. Thus,

$$\left(\frac{\partial S}{\partial \mathbf{x}}\right) = 2\sigma^{-1}\xi - 2\left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right)^T \Lambda,$$

and

$$\left(\frac{\partial S}{\partial \mathbf{a}}\right) = -2\left(\frac{\partial \mathbf{F}}{\partial \mathbf{a}}\right)^T \Lambda - 2\left(\frac{\partial \mathbf{G}}{\partial \mathbf{a}}\right)^T \Xi,$$

or

$$2\sigma^{-1}\xi - 2\mathbf{X}^T\Lambda = \mathbf{0}$$

and

$$-2\mathbf{A}^T\Lambda - 2\mathbf{D}^T\Xi = \mathbf{0}, \tag{3-7}$$

where the Jacobian matrices **X** and **A** are given by

$$\mathbf{X} = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right), \ \mathbf{A} = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{a}}\right), \ \mathbf{D} = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{a}}\right), \tag{3-8}$$

evaluated at the solutions. As σ is nonsingular, the first of equations (3-7) yields the important result

$$\xi = \sigma\mathbf{X}^T\Lambda, \tag{3–9}$$

which is needed for the computation of the adjustment corrections ξ. We use this equation to eliminate ξ from the first of equations (3-4), and can now write the set of general normal equations in the form

$$\begin{aligned} \mathbf{A}^T\Lambda + \mathbf{D}^T\Xi &= \mathbf{0} \\ \mathbf{F}(\mathbf{x}_0 + \sigma\mathbf{X}^T\Lambda, \mathbf{a}) &= \mathbf{0} \\ \mathbf{G}(\mathbf{a}) &= \mathbf{0} \end{aligned} \tag{3-10}$$

In many cases of physical and astronomical interest, there exist no exact relations among the parameters. In these instances, there are no equations $\mathbf{G(a)} = \mathbf{0}$ and the normal equations (3-10) reduce to

$$\begin{aligned} \mathbf{A}^T\Lambda &= \mathbf{0} \\ \mathbf{F}(\mathbf{x}_0 + \sigma\mathbf{X}^T\Lambda, \mathbf{a}) &= \mathbf{0}. \end{aligned} \tag{3-11}$$

Equations (3-10) and (3-11) are rigorous; no approximations have been made. But, as pointed out by Jefferys (1980), they generally form a set of unwieldy nonlinear equations, being linear only when the condition equations $\mathbf{F(x, a)} = \mathbf{0}$ are linear with respect to both the observations and the parameters. Note that, as these equations must be satisfied at the solution (when $\mathbf{a}$ and ξ have obtained their correct values), some iterative numerical approach must be employed in order to achieve a solution.

Solution of the Normal Equations

Both Jefferys (1980, 1981) and Eichhorn (1993) present schemes for solving equations (3-11). These authors develop approaches based on Newton's method, though Jefferys (1981) later presents techniques for solving the normal equations via the method of steepest descent, Marquardt's algorithm, or the Fletcher-Powell algorithm. All of these methods require an initial approximation, $\mathbf{a}_0$, to the parameter vector, $\mathbf{a}$, be available, with Newton's method more often than not requiring that the approximations be such that

$$\frac{a - a_0}{a} << 1 \tag{3-12}$$

in order for convergence to be achieved. Though often readily available, there do exist situations in which good parameter approximations are nonexistent; in these cases another technique, e.g., one of those outlined by Jefferys (1981) must be used. Despite the required closeness of the initial parameter approximations, a procedure for solving equations (3-11) by Newton's method shall now be presented, as it will serve to illustrate a basic approach to solving the nonlinear normal equations and also because it is sufficient for our purposes.

Let $\mathbf{a_0}$ denote the vector of approximations to the parameters, with α the corresponding vector of corrections to these parameters, i.e., $\mathbf{a} = \mathbf{a_0} + \alpha$. In a similar fashion, $\mathbf{x_0}$ represents the vector of observations and ξ is the associated vector of corrections to these observations. Thus

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_0 + \alpha \\ \mathbf{x} &= \mathbf{x}_0 + \xi. \end{aligned} \tag{3-13}$$

Equations (3-11), developed as a Taylor series and broken off after the first order terms, then become

$$\begin{aligned} \mathbf{F}_0 + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right)\bigg|_0 \xi + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{a}}\right)\bigg|_0 \alpha &= \mathbf{0} \\ \mathbf{A}^T \Lambda &= \mathbf{0}, \end{aligned}$$

or,

$$\begin{aligned} \mathbf{F}_0 + \mathbf{X}_0 \xi + \mathbf{A}_0 \alpha &= \mathbf{0} \\ \mathbf{A}_0{}^T \Lambda &= \mathbf{0}, \end{aligned} \tag{3-14}$$

where the subscripts mean that quantities in the condition equations and the partial derivatives are evaluated at $(\mathbf{a_0}, \mathbf{x_0})$. Note that $\mathbf{F_0} = \mathbf{F}(\mathbf{x_0}, \mathbf{a_0})$, which should be "small" if the observations were precise and equations (3-12) were satisfied. Substitution of the second of equations (3-13) into equation (3-9) yields

$$\xi_0 = \sigma \mathbf{X}_0^T \Lambda, \tag{3-15}$$

which may then be used in the first of equations (3-14), producing

$$\mathbf{F}_0 + \mathbf{X}_0 \sigma \mathbf{X}_0{}^T \Lambda + \mathbf{A}_0 \alpha = \mathbf{0}, \tag{3-16}$$

which may now be solved for Λ:

$$\Lambda = -\mathbf{W}_0 (\mathbf{F}_0 + \mathbf{A}_0 \alpha), \tag{3-17}$$

where

$$\mathbf{W}_0 = \left(\mathbf{X}_0 \sigma \mathbf{X}_0^T \right)^{-1} \tag{3-18}$$

The vector Λ obtained from equation (3-17) will not be the final Λ, as equation (3-16) is only an approximation. Call it therefore Λ_1, and find $\mathbf{x}_{\nu+1} = \mathbf{x}_\nu + \xi_\nu$, $\mathbf{a}_{\nu+1} = \mathbf{a}_\nu + \alpha_\nu$ ($\nu = 0$). Now form $\mathbf{X}_1$, $\mathbf{A}_1$ and $\mathbf{F}_1$, go through another iteration to find Λ_2, ξ_2 and α_2, and so forth until further iterations no longer produce any significant changes. Of course,

$$\xi = \sum_{\nu=0}^{N} \xi_\nu,$$

where we assume that the procedure has converged after N steps.

The elements of **W**, the "weight matrix", are well known to anyone who frequently adjusts data. It is important to note that one is not free to choose the weights arbitrarily, as has been done in many adjustments. Least squares requires that the observational weights be determined by the Jacobian matrix of the condition equations with respect to the observations and the covariance matrix of the observations; anything different will lead to biased parameter estimates. In the OLS adjustments performed by most scientists and engineers, **X** is an identity matrix. This results in **W** being diagonal, with the inverses of the observational variances comprising the elements. Therefore, the appropriate weights for an OLS adjustment are given by

$$W_{OLS_{i,i}} = \frac{1}{\sigma_i}, \tag{3-19}$$

where σ_i is the variance associated with the i-th observed quantity.

As with the observations, the parameter estimates are of limited usefulness unless they are accompanied by their covariance matrix. It can be shown that this matrix is given by

$$\sigma_{\alpha} = \left(\mathbf{A}^T \mathbf{W} \mathbf{A}\right)^{-1}. \tag{3-20}$$

The reader can find a derivation of equation (3-20) in most advanced statistics texts. Branham (1990) gives a simple development for linear systems, whereas Duane Brown (1955) and Jefferys (1980, 1981) present a more general treatment. The standard errors of the parameters are equal to the square roots of the diagonal elements of this matrix, or

$$\varepsilon_{\alpha_i} = \sqrt{\sigma_{\alpha_{ii}}} \tag{3-21}$$

A Simple Example

The matrix notation used in presenting data adjustment schemes is similar to the terminology employed by astrometrists; both seem to contribute to the confusion of the novice, even though the concepts are actually quite simple. It would therefore seem appropriate to illustrate the general least-squares procedure outlined in the previous section by use of a physical example, one that has somewhat of a bearing on the Hyades' distance.

As mentioned in chapter 2, the Wilson-Bappu effect states that there is a linear relationship between the logarithm of the CaII emission line width and the absolute visual magnitude of the star. We may therefore write

$$M_{\mathrm{v}} - q\log W + b = 0, \tag{3-22}$$

where M_V is the star's indirectly observed absolute visual magnitude, W is the observed width of the emission line, and q and b are the unknown parameters. The equation above, when set up for each of the stars involved in the adjustment, form the set of condition equations

$$\mathbf{F} = \begin{pmatrix} M_{\mathrm{v}_1} - q\log W_1 + b \\ M_{\mathrm{v}_2} - q\log W_2 + b \\ \vdots \\ M_{\mathrm{v}_m} - q\log W_m + b \end{pmatrix} = \mathbf{0}, \tag{3-23}$$

where the subscripts indicate the particular star considered, m being the total number of stars. These quantities are considered to be the observations, resulting in

$$\mathbf{x}_{\mathrm{obs}}^T = \begin{pmatrix} M_{\mathrm{v}_1} & W_1 & M_{\mathrm{v}_2} & W_2 & \cdots & M_{\mathrm{v}_m} & W_m \end{pmatrix} \tag{3-24}$$

Note that each star generates two "observations" and that all these are incorporated into one vector. Accompanying each M_V and W should be the associated standard error, denoted by ε_M and ε_W, respectively. Assuming the observations to be uncorrelated, let the covariance matrix of these observations be given by

$$\sigma = \begin{pmatrix} \varepsilon_{M_1}^2 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_{W_1}^2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \varepsilon_{M_m}^2 & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon_{W_m}^2 \end{pmatrix} \tag{3-25}$$

We now set up the Jacobian matrices **A** and **X** by taking the partial derivatives of equations (3-28) with respect to the parameters and the observations. This results in

$$\mathbf{A} = \begin{pmatrix} -\log W_1 & 1 \\ -\log W_2 & 1 \\ \vdots & \vdots \\ -\log W_m & 1 \end{pmatrix}, \text{ and} \tag{3-26}$$

$$\mathbf{X} = \begin{pmatrix} 1 & -\dfrac{q\ln(10)}{W_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\dfrac{q\ln(10)}{W_2} & & 0 & 0 \\ & & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\dfrac{q\ln(10)}{W_m} \end{pmatrix} \tag{3-27}$$

as the parameter vector, $\mathbf{a}_0$, is given by

$$\mathbf{a}_0^T = \begin{pmatrix} q & b \end{pmatrix} \tag{3-28}$$

Having constructed the matrices needed to perform the adjustment, the parameter estimates can be obtained by proceeding as follows:

1) Start with $\nu = 0$, i.e., $\mathbf{x}_0$, and assume a plausible $\mathbf{a}_0^T = (q_0 \quad b_0)$, where q_0 and b_0 are initial guesses for the parameters, obtained from Eggen (1967). Set up the diagonal covariance matrix $\boldsymbol{\sigma}$ from the observational errors.

2) Compute $\mathbf{A}_\nu$, $\mathbf{X}_\nu$, $\mathbf{W}_\nu$, and $\mathbf{F}_\nu$ from the current values of $\mathbf{x}_\nu$ and $\mathbf{a}_\nu$.

3) Calculate the vector of the parameter corrections, $\boldsymbol{\alpha}_\nu$, from

$$\begin{aligned} \alpha &= -\left(\mathbf{A}_\nu^T \mathbf{W}_\nu \mathbf{A}_\nu\right)^{-1} \mathbf{A}_\nu^T \mathbf{W}_\nu F_\nu \\ \Lambda_\nu &= -\mathbf{W}_\nu\left(\mathbf{F}_\nu + \mathbf{A}_\nu \alpha_\nu\right) \\ \xi_\nu &= \sigma \mathbf{X}_\nu^T \Lambda_\nu \end{aligned}$$

4) Compute the new quantities $\mathbf{x}_{v+1}$, $\mathbf{a}_{v+1}$ and $\mathbf{F}_{v+1}$.

5) If the components of $\mathbf{F}_{v+1}$ are sufficiently small, go to step 6) below. Otherwise, go to step 2).

6) Convergence has been achieved; compute the errors in the parameters by using

$$\sigma_{\boldsymbol{\alpha}} = \left(\mathbf{A}^T \mathbf{W} \mathbf{A}\right)^{-1}$$
$$\varepsilon_{\alpha_i} = \sqrt{\sigma_{\alpha_{ii}}}$$

It would seem from the above that a general least-squares solution is not at all difficult to implement, requiring slightly more work than that involved in an ordinary least-squares approach. Conceptually, this is indeed true; however, the implementation of a general least-squares adjustment requires considerably more planning and far more computer power than the corresponding OLS technique. The reason for this lies in the large size of the matrices associated with the observations. In the simple example presented, **X** is of the order of m x 2m, whereas **A** is only m x 2 in size. An adjustment involving a more elaborate set of condition equations and many observations can easily produce a matrix **X** with over a million elements, making $\mathbf{X}\sigma\mathbf{X}^T$ of similar size. This product must be inverted in order to obtain the "weight" matrix, **W**, a herculean task for even modern computers. Fortunately, **X** is "sparse", that is, most of its elements are 0. This results in $\mathbf{X}\sigma\mathbf{X}^T$ being block diagonal and symmetrical, enabling the computation of **W** by the inversion of the much smaller individual blocks via efficient inversion routines. There also now exist matrix routines (e.g., Branham, 1990) that can greatly facilitate multiplications and other operations involving sparse matrices, and these should be seriously considered for inclusion in the adjustment program.

Probabilistic Constraints

Eichhorn (1978) has further extended the theory of least squares to enable the estimation of parameters which are samples of a Gaussian distribution with known covariance but which whose modes (i.e., average values) are unknowns. There are numerous astrometric instances where such a technique would be valuable, one example being the restriction of certain parameters in an overlap solution incorporating photographic plates taken with the same instrument. In this instance, the focal length of the optical system can be allowed to vary about some initially undetermined mean value, resulting in more accurate star positions than would be the case if the focal length were unrestricted (a "free" parameter) or forced to have the same value for all plates. Another example involves restricting the velocities and parallaxes (distances) of the member stars of an open cluster to obey Gaussian distributions with known variances, and is the focus of this work.

Least squares with probabilistic constraints assumes the presence of two parameter types, the first of which is that vector of unconstrained, "conventional" parameters familiar to least-squares practitioners. The second type of parameter is the vector of parameters subject to probabilistic constraints; hence the technique's name, "least-squares with probabilistic constraints". This vector of constrained parameters really consists of two parts; the vector **g** of the mode and the random vector $\boldsymbol{\beta}$ of the individual variations. We assume that we have an approximation $\mathbf{g}_0^T$ to $\mathbf{g}^T = \begin{pmatrix} g_1 & g_2 & \cdots & g_k \end{pmatrix}$ and thus need to find the *k*-vector $\boldsymbol{\gamma}$ of corrections, implying that we have put

$$\mathbf{g} = \mathbf{g}_0 + \boldsymbol{\gamma}. \tag{3-29}$$

In addition to each component g_κ of **g** belongs a μ_κ-vector $\boldsymbol{\beta}_\kappa^T = \begin{pmatrix} \beta_{\kappa 1} & \beta_{\kappa 2} & \cdots & \beta_{\kappa \mu_\kappa} \end{pmatrix}$, representing the μ_κ randomly distributed samples $\boldsymbol{\beta}_\kappa$ of the component g_κ of the vector **g**. We combine all the μ_κ-vectors $\boldsymbol{\beta}_\kappa$ into the $\sum_{\kappa=1}^{k} \mu_\kappa$-vector $\boldsymbol{\beta}$, i.e.,

$$\boldsymbol{\beta}^T = \left(\boldsymbol{\beta}_1^T \quad \boldsymbol{\beta}_2^T \quad \cdots \quad \boldsymbol{\beta}_k^T\right)$$

and introduce the $\sum_{\kappa=1}^{k} \mu_\kappa$-vector **b** by

$$\mathbf{b}^T = \left(\mathbf{e}_1 g_1 + \boldsymbol{\beta}_1 \quad \cdots \quad \mathbf{e}_k g_k + \boldsymbol{\beta}_k\right), \qquad (3\text{-}30)$$

where the μ_κ-vector $\mathbf{e}_\kappa$ is given by

$$\mathbf{e}_\kappa^T = \left(1 \quad \cdots \quad 1\right).$$

We assume that the $\sum_{\kappa=1}^{k} \mu_\kappa$ components of $\boldsymbol{\beta}$ obey the distribution function (Eichhorn, 1978),

$$\psi(\boldsymbol{\beta}) = Ke^{-\frac{1}{2}\boldsymbol{\beta}^T \boldsymbol{\rho}^{-1} \boldsymbol{\beta}} \qquad (3\text{-}31)$$

or some other exponential function that shares the same maximum.

Let the equations of condition be given by

$$\mathbf{F}[\mathbf{x}, \mathbf{b}(\mathbf{g}, \boldsymbol{\beta}), \mathbf{a}] = \mathbf{0} \qquad (3\text{-}32)$$

where, as before, $\mathbf{a} = \mathbf{a}_0 + \boldsymbol{\alpha}$. The joint distribution function of the observations and the restricted parameters is then

$$\psi(\boldsymbol{\beta})\varphi(\boldsymbol{\xi}) = Ke^{-\frac{1}{2}\boldsymbol{\beta}^T \boldsymbol{\rho}^{-1} \boldsymbol{\beta}} e^{\frac{-1}{2}\boldsymbol{\xi}^T \boldsymbol{\sigma}^{-1} \boldsymbol{\xi}} = Ke^{-\frac{1}{2}\left(\boldsymbol{\xi}^T \boldsymbol{\sigma}^{-1} \boldsymbol{\xi} + \boldsymbol{\beta}^T \boldsymbol{\rho}^{-1} \boldsymbol{\beta}\right)}\text{[5]} \qquad (3\text{-}33)$$

We wish to solve equations (3-32) for **a** and **g** while simultaneously minimizing

$$S = \boldsymbol{\xi}^T \boldsymbol{\sigma}^{-1} \boldsymbol{\xi} + \boldsymbol{\beta}^T \boldsymbol{\rho}^{-1} \boldsymbol{\beta} \qquad (3\text{-}34)$$

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[5] Note that here we have "sneaked" in the assumption that both distribution functions are Gaussian.

As before, the technique of Lagrange multipliers may be used, resulting in

$$S = \xi^T \sigma^{-1} \xi + \beta^T \rho^{-1} \beta - 2\Lambda^T \left(\mathbf{X}\xi + \mathbf{A}\alpha + \mathbf{B}\beta + \mathbf{C}\gamma + \mathbf{F}_0 \right), \tag{3-35}$$

where **B** is the Jacobian matrix of partial derivatives of **F** with respect to the probabilistically restricted parameter corrections, β, and **C** is the matrix of partial derivatives of **F** with respect to the means of the restricted parameters, **g**, i.e.,

$$\mathbf{B} = \left(\frac{\partial \mathbf{F}}{\partial \beta} \right), \quad \mathbf{C} = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}} \right) \tag{3-36}$$

Note that the matrices **B** and **C** are built up of the same components but that the dimension of **C** is m x k, while that of **B** is m x $\sum_{\kappa=1}^{k} \mu_\kappa$. Taking the partial derivatives of the above and setting them equal to 0 gives the normal equations

$$\begin{aligned} \Lambda^T \mathbf{A} &= \mathbf{0} \\ \Lambda^T \mathbf{C} &= \mathbf{0} \\ \mathbf{F}(\mathbf{x}, \mathbf{a}, \mathbf{g}) &= \mathbf{0}, \end{aligned} \tag{3-37}$$

in which ξ and β have been expressed in terms of Λ by the

$$\begin{aligned} \xi &= \sigma \mathbf{X}^T \Lambda \\ \beta &= \rho \mathbf{B}^T \Lambda \end{aligned} \tag{3-38}$$

Solving the system for Λ yields

$$\Lambda = -\mathbf{W}(\mathbf{A}_0 \alpha + \mathbf{C}_0 \gamma + \mathbf{F}_0), \tag{3-39}$$

where the weight matrix **W** is now given by

$$\mathbf{W} = \left(\mathbf{X}_0 \sigma \mathbf{X}_0^T + \mathbf{B}_0 \rho \mathbf{B}_0^T \right)^{-1} \tag{3-40}$$

Equation (3-40) is a beautiful illustration of the similarity between the observations and the restricted parameters. Both follow Gaussian distributions, both their covariance matrices must be known, and both enter into the mathematics of the adjustment in like fashion.

The parameter corrections, α and γ, may be obtained by solving the system

$$\begin{pmatrix} \mathbf{A}_0^T\mathbf{W}\mathbf{A}_0 & \mathbf{A}_0^T\mathbf{W}\mathbf{C}_0 \\ \mathbf{C}_0^T\mathbf{W}\mathbf{A}_0 & \mathbf{C}_0^T\mathbf{W}\mathbf{C}_0 \end{pmatrix}\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}+\begin{pmatrix} \mathbf{A}_0^T \\ \mathbf{C}_0^T \end{pmatrix}\mathbf{W}\mathbf{F}_0=\mathbf{0} \tag{3-41}$$

As the normal equations will be generally nonlinear, a numerical approach similar to that outlined in the previous section must be used to obtain a solution. **a**, **g** and **x** are updated according to

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_0+\alpha \\ \mathbf{g} &= \mathbf{g}_0+\gamma \\ \mathbf{x} &= \mathbf{x}_0+\xi, \end{aligned} \tag{3-42}$$

where ξ is given by

$$\xi=-\sigma\mathbf{X}^T\mathbf{W}\left(\mathbf{A}_0\alpha+\mathbf{C}_0\gamma+\mathbf{F}_0\right) \tag{3-43}$$

Once convergence has been achieved, the vector β is calculated from

$$\beta=-\rho B^T\mathbf{W}\left(\mathbf{A}_0\alpha+\mathbf{C}_0\gamma+\mathbf{F}_0\right), \tag{3-44}$$

and then the restricted parameters from equation (3-30). Finally, the covariance matrix of the parameters is obtained from the analogy to equation (3-20).

The Condition Number and the Efficiency

Occasionally a practitioner of least squares will hear some matrix described as "ill-conditioned". This has nothing to do with the everyday conventional definition of "condition", as a matrix has no health; rather, it refers to the suitability of the matrix for inversion. This suitability is measured by the "condition number," k, of the matrix, defined for symmetric positive definite matrices by

$$k = \frac{\lambda_{\max}}{\lambda_{\min}}, \tag{3-45}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the matrix. The larger the condition number, the less accurate the inverse. This is because k is essentially the decadic logarithm of the factor by which the inaccuracies in the elements of the matrix are multiplied when calculating the inverse (see Branham, 1990 for more details). For example, a k of the order of 10^6 would indicate that a matrix could not be inverted using single precision arithmetic, which has a precision of 10^{-6}.

The efficiency, e, of a symmetric, positive definite matrix, **Q**, defined by Eichhorn (1993) as

$$e = \sqrt[n]{\frac{|\mathbf{Q}|}{\prod_{i=1}^{n} Q_{i,i}}} \tag{3-50}$$

where n is the rank of **Q**, measures still another property of the matrix. If **Q** is the covariance matrix of a vector **a**, the efficiency is a measure of the degree of correlation between the elements of **a**. A low efficiency indicates a high degree of correlation, originating from the off-diagonal elements of **Q**, whereas a high efficiency (e ~ 1) indicates a low degree of correlation between the elements of **a**. The application of this to least-squares adjustments should be obvious, as the efficiency of the covariance matrix of the parameters will indicate the degree of correlation among the parameters. It is important to note that the efficiency and the condition number are independent of each other.

CHAPTER 4
THE HYADES DISTANCE

The Normal Equations for the Hyades

Having outlined the theory behind least squares adjustments with probabilistic constraints in chapter 3, it is now time to focus on the application of this theory to the problem of the determination of the distances and velocities of the stars belonging to the Hyades cluster[1]. The first step towards this problem's solution is the same as the initial step in any adjustment procedure, namely, to obtain the observations and set up the equations of condition. The latter has already been done, for in chapter 2 we saw that for any star,

$$\begin{pmatrix} \pi\dot{r} \\ \dot{\alpha}\cos\delta \\ \dot{\delta} \end{pmatrix} = \pi\mathbf{R}_3(-\delta)\mathbf{R}_2(\alpha)\dot{\mathbf{x}}, \text{ or}$$

$$\begin{pmatrix} \dot{r} - \dot{x}\cos\alpha\cos\delta - \dot{y}\sin\alpha\cos\delta - \dot{z}\sin\delta \\ \kappa\mu_\alpha + \dot{x}\pi\sin\alpha - \dot{y}\pi\cos\alpha \\ \kappa\mu_\delta + \dot{x}\pi\cos\alpha\sin\delta + \dot{y}\pi\sin\alpha\sin\delta - \dot{z}\pi\cos\delta \end{pmatrix} = \mathbf{0}, \qquad (4\text{-}1)$$

where we have replaced the proper motion components $\dot{\alpha}\cos\delta$ and $\dot{\delta}$ by μ_α and μ_δ, the units being the only difference. As before, $\dot{r}$ is the star's radial velocity, α, δ are the components of its position, π is its parallax (in " A.U.$^{-1}$), and $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the stellar equatorial velocity components in the same units as $\dot{r}$. The set of equations (4-1) involving all stars used in the adjustment shall form the condition equations, **F**. Therefore,

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[1] Indeed, in the first paper published on least squares with probabilistic constraints, Eichhorn (1978) suggests the application of this technique to open clusters.

$$\mathbf{F}=\begin{pmatrix} \dot{r}_1-\dot{x}\cos\alpha_1\cos\delta_1-\dot{y}\sin\alpha_1\cos\delta_1-\dot{z}\sin\delta_1 \\ \kappa\mu_{\alpha 1}+\dot{x}\pi_1\sin\alpha_1-\dot{y}\pi_1\cos\alpha_1 \\ \kappa\mu_{\delta 1}+\dot{x}\pi_1\cos\alpha_1\sin\delta_1+\dot{y}\pi_1\sin\alpha_1\sin\delta_1-\dot{z}\pi_1\cos\delta_1 \\ \vdots \\ \dot{r}_m-\dot{x}\cos\alpha_m\cos\delta_m-\dot{y}\sin\alpha_m\cos\delta_m-\dot{z}\sin\delta_m \\ \kappa\mu_{\alpha m}+\dot{x}\pi_m\sin\alpha_m-\dot{y}\pi_m\cos\alpha_m \\ \kappa\mu_{\delta m}+\dot{x}\pi_m\cos\alpha_m\sin\delta_m+\dot{y}\pi_m\sin\alpha_m\sin\delta_m-\dot{z}\pi_m\cos\delta_m \end{pmatrix}=\mathbf{0} \tag{4-2}$$

where m is the number of stars.

Obviously, the observations for each star are its proper motion components, radial velocity and parallax. In most other astrometric circumstances, the position (α, δ) is also regarded as a pair of observations; however, in this case the positions are known with such great precision (compared to the errors in the other observables) that they may treated as known parameters and not subjected to adjustment. It should be pointed out that, though the Hyades have been an object of study for many decades, a full set of all four observables $\mathbf{x}_{\mathbf{0}_i}^{\mathbf{T}} = (\dot{r}_i\ \mu_{\alpha_i}\ \mu_{\delta_i}\ \pi)$, is available for only a small number of the stars in the cluster. Fortunately, the adjustment procedure can accommodate even the minimum set $\mathbf{x}_{\mathbf{0}_i}^{\mathbf{T}} = (\mu_{\alpha_i}\ \mu_{\delta_i})$, by simply leaving out those components of **F** which involve the star's radial velocity, i.e., the first of equations (4-2). In the discussion to follow, it will simplify matters to assume that all four observables are available for each star, so that the vector of observations is

$$\mathbf{x}_{\mathbf{0}}^{T}=\begin{pmatrix}\dot{r}_1 & \mu_{\alpha_1} & \mu_{\delta_1} & \pi_1 & \cdots & \dot{r}_m & \mu_{\alpha_m} & \mu_{\delta_m} & \pi_m\end{pmatrix}, \tag{4-3}$$

with associated covariance matrix, σ:

$$\sigma=\begin{pmatrix} \varepsilon_{\dot{r}_1}^2 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_{\mu_{\alpha_1}}^2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \varepsilon_{\mu_{\delta_m}}^2 & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon_{\pi_m}^2 \end{pmatrix}, \tag{4-4}$$

where the ε denote standard errors in the appropriate quantities. The above assumes that the observations are uncorrelated; correlated observations will have non-zero off-diagonal elements. Note that the observation vector is of length *4m* and that the sparse observation covariance matrix has dimensions *4m* x *4m*.

The unknown parameters in equations (4-2) are the three cartesian velocity components and the parallaxes of the stars, the parallaxes entering into the adjustment as both observations and parameters. All parameters shall be stochastically (probabilistically) constrained, the reasons for their restriction being set forth by Eichhorn:

> It is inaccurate to assume that all stars have the same velocity. Their velocities will rather obey a certain distribution function, for which an (if necessary, trivariate) Maxwellian distribution function would be a good approximation. . . . The estimation of the parallaxes of the member stars of a moving cluster poses a similar problem. If these are estimated individually without any restrictions, they will have more or less large errors and thus the cluster will seem more elongated in the line of sight than it actually is. (Eichhorn , 1978, p. 355-356)[2]

Therefore the vector of conventional unrestricted parameters, **a**, does not exist in our problem, and we may write

$$\begin{aligned} \mathbf{g}^T &= \left(\langle\dot{x}\rangle \quad \langle\dot{y}\rangle \quad \langle\dot{z}\rangle \quad \langle\pi\rangle \quad \cdots \quad \langle\dot{x}\rangle \quad \langle\dot{y}\rangle \quad \langle\dot{z}\rangle \quad \langle\pi\rangle\right) \\ &\quad +\left(\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \cdots \quad \beta_{4m-3} \quad \beta_{4m-2} \quad \beta_{4m-1} \quad \beta_{4m}\right), \end{aligned} \tag{4-5}$$

---

[2] Note that in fact, the velocity distribution function cannot be Maxwellian, because the velocity of a star at the cluster's boundary cannot have a radial component (with respect to the cluster center). Nevertheless, it is better to assume a Maxwellian distribution than to constrain the velocity to be the same for each star.

where the β's are the stochastic parameters, and the expectation values of the parameters are given by

$$\langle \dot{x} \rangle = \dot{x}_0 + \gamma_1,\ \langle \dot{y} \rangle = \dot{y}_0 + \gamma_2,\ \langle \dot{z} \rangle = \dot{z}_0 + \gamma_3,\ \langle \pi \rangle = \pi_0 + \gamma_4,$$

the γ's being the parameter corrections. The expectation values yield the average cluster velocity and parallax; these are added to the stochastic β's (which have expectation value zero) to yield the parameters for each star. γ is a 4-vector, whereas β is a *4m* vector; hence the form of equation (4-5). Before attempting the adjustment, reasonable values for the elements of the covariance matrix, ρ, of the constrained parameters must be determined by some means. As the parameters are assumed to be uncorrelated, this matrix is of the form

$$\rho = \begin{pmatrix} \varepsilon_{1\dot{x}}^2 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_{1\dot{y}}^2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \varepsilon_{m\dot{z}}^2 & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon_{\pi}^2 \end{pmatrix}, \tag{4-6}$$

where the ε's refer to the dispersions in the indicated parameters. It should be pointed out that the dispersions in the cartesian velocity components can be assumed equal, and that the value of this dispersion can, at least in principle, be obtained from a study of the radial velocities of the stars which are Hyades members. The dispersion in the parallaxes can be obtained from the angular size of the cluster or from dynamical theory.

We now set up the various Jacobian matrices required for the adjustment. An application of equations (3-8) and (3-40) to the equations of condition results in the matrices

$$\mathbf{C}=\begin{pmatrix}-\cos\alpha_1\cos\delta_1 & -\sin\alpha_1\cos\delta_1 & -\sin\delta_1 & 0\\ \pi\sin\alpha_1 & -\pi\cos\alpha_1 & 0 & \dot{x}\sin\alpha_1-\dot{y}\cos\alpha_1\\ \pi\cos\alpha_1\sin\delta_1 & \pi\sin\alpha_1\sin\delta_1 & -\pi\cos\delta_1 & \dot{x}\cos\alpha_1\sin\delta_1+\dot{y}\sin\alpha_1\sin\delta_1-\dot{z}\cos\delta_1\\ & \vdots & & \vdots\\ \pi\cos\alpha_m\sin\delta_m & \pi\sin\alpha_m\sin\delta_m & -\pi\cos\delta_m & \dot{x}\cos\alpha_m\sin\delta_m+\dot{y}\sin\alpha_m\sin\delta_m-\dot{z}\cos\delta_m\end{pmatrix} \quad (4\text{-}7)$$

$$\mathbf{X}=\begin{pmatrix}1\ 0\ 0 & 0 & & 0\\ 0\ 1\ 0 & \dot{x}\sin\alpha_1-\dot{y}\cos\alpha_1 & \cdots & 0\\ 0\ 0\ 1 & \dot{x}\cos\alpha_1\sin\delta_1+\dot{y}\sin\alpha_1\sin\delta_1-\dot{z}\cos\delta_1 & & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0\ 0\ 0 & 0 & & 0\\ 0\ 0\ 0 & 0 & \cdots & \dot{x}\sin\alpha_m-\dot{y}\cos\alpha_m\\ 0\ 0\ 0 & 0 & & \dot{x}\cos\alpha_m\sin\delta_m+\dot{y}\sin\alpha_m\sin\delta_m-\dot{z}\cos\delta_m\end{pmatrix}$$

Note that **X** is sparse and block diagonal, each block being *3* x *4* in the case of a full complement of observations for the star, giving **X** a maximum size of *3m* x *4m*. If the parallax is not available, the block reduces to a *3* x *3* identity matrix, and, if only proper motions have been observed for the star, the block is smaller still (a *2* x *2* identity matrix). The lack of an observed parallax has no influence on **C** (which has maximum dimensions of *3m* x *4*), though a star with only proper motions available will have just two of the three rows, as the first of equations (4-1) must be dropped. Equations (3-40) also establish the matrix **B** as

$$\mathbf{C}_i=\begin{pmatrix}-\cos\alpha_i\cos\delta_i & -\sin\alpha_i\cos\delta_i & -\sin\delta_i & 0\\ \pi\sin\alpha_i & -\pi\cos\alpha_i & 0 & \dot{x}\sin\alpha_i-\dot{y}\cos\alpha_i\\ \pi\cos\alpha_i\sin\delta_i & \pi\sin\alpha_i\sin\delta_i & -\pi\cos\delta_i & \dot{x}\cos\alpha_i\sin\delta_i+\dot{y}\sin\alpha_i\sin\delta_i-\dot{z}\cos\delta_i\end{pmatrix}$$

$$\mathbf{B}=\begin{pmatrix}\mathbf{C}_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_2 & & \mathbf{0}\\ \vdots & & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_m\end{pmatrix} \qquad (4\text{-}8)$$

There being no conventional, unconstrained parameters, the solved normal equations (3-41) reduce to

$$\boldsymbol{\gamma} = -\left(\mathbf{C}^T\mathbf{W}\mathbf{C}\right)^{-1}\mathbf{C}^T\mathbf{W}\mathbf{F}_0, \tag{4-9}$$

where $\mathbf{W}$ and $\mathbf{F}_0$ are defined by equations (3-18) and (3-10). An iterative procedure similar to that outlined in chapter 3 is then used to obtain the vector $\boldsymbol{\gamma}$. Once convergence has been reached, the parameter covariance matrix is found from equation (3-25) and the vector $\boldsymbol{\beta}$ calculated from equation (3-44). Finally, the stochastically restricted parameters (the velocity components and parallaxes of each star) are computed from equation (4-5).

Application of the Technique to Model Clusters

Description of the Models

One of the truly wonderful aspects of modern electronic computers is their capability to realistically simulate many physical situations, including the motions of stars which are members of open clusters. This makes it possible to construct a hypothetical cluster and generate "observations" of its member stars, which may then be used in various adjustment procedures to test the accuracies of their parameter estimates. Nature does not afford us this luxury, as astrometrists have no way of knowing the true velocities and locations of stars in an open cluster. The introduction of a new technique naturally raises questions as to its ability to determine more accurate parameter estimates than those obtained by previous methods, questions that cannot be easily resolved by application to the real world. Therefore, it would seem advisable to utilize some of the available computing power to generate model clusters to serve as tests for the method of least squares with probabilistic constraints.

There is no need to perform lengthy n-body calculations to generate the models, as the important consideration is that the velocity dispersion of the "stars" should be small compared to the average cluster space velocity. The adopted equations of condition do not require that a star be gravitationally bound to the other stars in the cluster. This simplification makes for rapid generation of models with a large number of stars, even on "slow" computers.

Twenty-four model "associations" of 100 stars each were constructed and placed into three distinct sets. The first set consisted of ten models having centers located at a distance of 50 parsecs (parallax of 0".02) at equatorial position (60°, 15°). The distances of the individual stars from the cluster centers were allowed to vary according to

$$r_c = 5\,\mathrm{rnd}(0,\ 1),$$

where r_c is the distance in parsecs of the star from the cluster center, and rnd (0, 1) represents a Gaussian random number between 0 and 1. A distribution of distances from the cluster center belonging to one of the models is displayed in figure 4-1, and figure 4-2 is a plot of the stars' equatorial positions.

The average space velocity for these ten association models has the components $v^T = (-1.4, 44.5,\ 6.5)$ km s^{-1} and thus a speed of 45 km s^{-1}. This velocity vector yields a convergent "point" position at (91°.8, 8°.3). As with the distances, the velocity components of the individual stars were allowed to deviate from the average values according to a Gaussian distribution, the differences being given by

$$\begin{aligned} v_{x_c} &= 0.5\ \mathrm{rnd}\ (-1,1) \\ v_{yc} &= 0.5\ \mathrm{rnd}\ (-1,1), \\ v_{z_c} &= 0.5\ \mathrm{rnd}\ (-1,1) \end{aligned}$$

where rnd (-1,1) represents a Gaussian random number between -1 and 1.

Next, the "observations" were calculated by means of equations (2-14). These observations were then "dirtied up" by the addition of synthetic errors produced by Gaussian random number generators. The width of the Gaussian used to generate the proper motion errors was set at 0.005 " yr^{-1}, with corresponding values of 0.5 km s^{-1} and

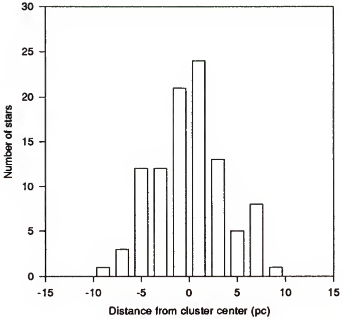

Figure 4-1. Distribution of stellar distances from cluster center.

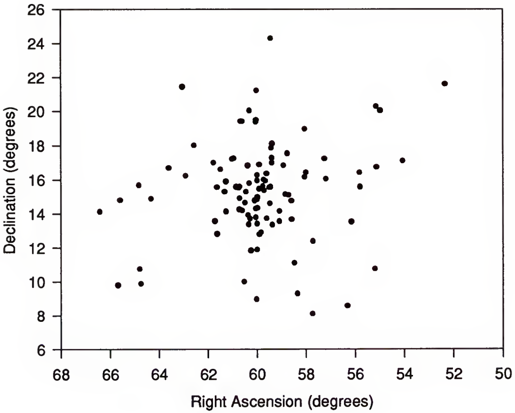

Figure 4-2. Equatorial positions of association stars (Gaussian distribution).

0".01 for the radial velocity and parallax errors, respectively. The effects of these errors upon the proper motions and radial velocities is minor; this is not the case with the parallax "observations", as the standard deviation of 0".01 is one-half the value of the mean parallax. Figure 4-3 depicts the number distribution of the parallax "observations" after the errors have been applied; note the huge scatter in the parallaxes, some of which have even become negative.

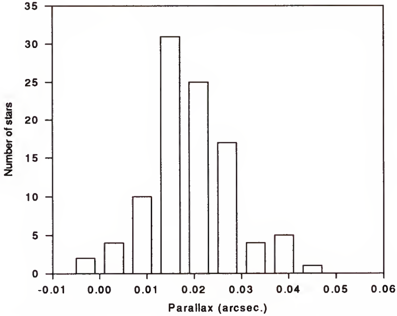

Figure 4-3. Frequency distribution of synthetic parallax "observations".

In order to test the sensitivity of the adjustment procedure to the distance and velocity distributions of the stars within a cluster, the second group of ten model associations was generated with the same center location and average velocity as those of the first set. However, the distances of the stars from the cluster center and the velocities were chosen at random from a uniform distribution, rather than from a Gaussian. In order to keep consistency with the previous group, the maximum distance from the cluster center was set at 5 parsecs and the velocity components were allowed to deviate from the average value by no more than 0.5 km s^{-1}. Synthetic "observations" were then produced and

Gaussian errors applied in the fashion described for the first set of associations. Figure 4-4 is a depiction of the equatorial positions of the stars belonging to one model in this second group of associations; note that the cluster is more compact than its counterparts from the Gaussian set.

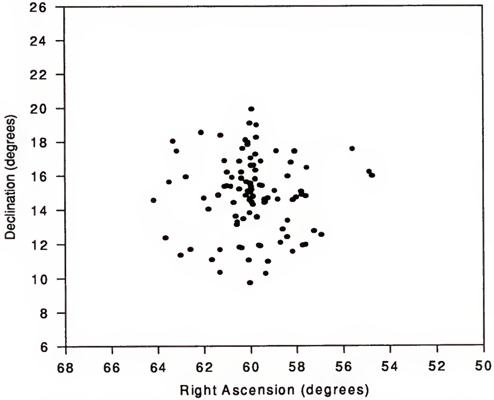

Figure 4-4. Equatorial positions of association stars (uniform distribution).

The third set of models consisted of four associations constructed in a fashion identical to those in the first group, save that the position of the cluster centers was moved from (60°, 15°) to (110°, 45°). These associations also had their average velocity altered to $v^T = (-38.4, 6.8, 22.5)$ km s^{-1}, yielding a convergent "point" at equatorial position (170°, 30°) and a λ of 48º.7 for the cluster center. The primary purpose for the creation of this smaller third group was to make sure that the adjustment techniques were not sensitive to the direction of the cluster's velocity (and, consequently, to the position of the convergent point) and to its position in the sky.

Evaluation of the Adjustment Procedures

Using the equations and procedures outlined in chapter 3 and in the first part of this chapter, least-squares adjustments involving probabilistic constraints were performed on all model associations, with the parallax dispersion set at 0".006 and the dispersions in the velocity components set equal to 0.5 km s^{-1}. The global parameter estimates for the Gaussian association models are summarized in table 4-1. It is readily apparent that there is some scatter in the average association distance and velocity; however, the averages of these estimates are comfortably close to their true values. Also note that no model has global parameters that deviate from the true values by more than the quoted standard error.

Table 4-1 is reassuring in that it presents evidence that an adjustment involving stochastic constraints can reliably characterize the gross characteristics of an association or cluster. However, the thrust of the stochastic constraint technique is that the velocities and parallaxes of the individual stars can be directly determined without the errors in the observed quantities exercising their full distorting influence, and it is essential to see how closely these estimates conform to reality. Some indication can be gleaned from table 4-2, which gives the average of the absolute deviations of the stochastic velocities and parallaxes from the stars' true values. It would seem that we can match a star's velocity to within 1 km s^{-1}, and the parallax to about 0".001, which is the quoted precision of the MAP parallaxes (Gatewood et al., 1992). For a mean distance of 50 parsecs, an error of 0".001 in the parallax would give a corresponding distance error of about 2.5 parsecs. This does not seem very impressive; however, as stated previously, these adjustments were performed to compare to the results of the adjustments involving the uniformly-distributed model associations. Therefore, the constraint in the parallax was deliberately set to a very loose value of 0".006 (15 parsecs). It shall be seen later that an appropriate "tightening" of the parallax restriction can result in an average absolute

Table 4-1. Global parameters for Gaussian association models

Model No.	v_{xc} (km s^{-1})	v_{yc} (km s^{-1})	v_{zc} (km s^{-1})	π_c (arcsec.)
1	-1.29 ± 1.17	44.58 ± 0.76	6.47 ± 0.25	0.0199 ± 0.0017
2	-0.41 ± 1.18	43.96 ± 0.77	6.48 ± 0.27	0.0209 ± 0.0018
3	0.36 ± 1.09	43.49 ± 0.69	6.7 ± 0.22	0.0217 ± 0.0017
4	-1.85 ± 1.2	44.94 ± 0.78	6.39 ± 0.24	0.0195 ± 0.0016
5	-2.71 ± 1.35	45.42 ± 0.85	6.34 ± 0.28	0.0186 ± 0.0017
6	-0.93 ± 1.13	44.29 ± 0.74	6.56 ± 0.23	0.0203 ± 0.0017
7	-2.96 ± 1.26	45.54 ± 0.81	6.31 ± 0.25	0.0183 ± 0.0016
8	-0.9 ± 1.13	44.3 ± 0.73	6.43 ± 0.25	0.0204 ± 0.0017
9	-2.38 ± 1.16	45.25 ± 0.74	6.49 ± 0.24	0.0188 ± 0.0016
10	-0.74 ± 1.18	44.16 ± 0.74	6.5 ± 0.22	0.0207 ± 0.0017
Average	-1.38 ± 1.07	44.59 ± 0.68	6.47 ± 0.11	0.0199 ± 0.0011

Table 4-2. Deviations of stochastic parameters from true values for Gaussian models.

Model No.	$\lvert \Delta v_x \rvert$ (km s^{-1})	$\lvert \Delta v_y \rvert$ (km s^{-1})	$\lvert \Delta v_z \rvert$ (km s^{-1})	$\lvert \Delta\pi \rvert$ (mas*)
1	0.14	0.20	0.14	0.7
2	0.62	0.36	0.14	0.8
3	1.77	1.04	0.24	1.8
4	0.44	0.42	0.16	1.0
5	1.30	0.91	0.18	1.5
6	0.48	0.31	0.14	0.9
7	1.55	1.03	0.21	1.6
8	0.51	0.28	0.15	0.8
9	0.97	0.74	0.14	1.2
10	0.67	0.40	0.14	0.9
Average	0.84	0.57	0.16	1.1

* mas is the unit abbreviation for 0".001.

deviation in the parallax of 0".0005, an improvement by a factor of two over the value given in table 4-2.

Tables 4-3 and 4-4 present similar results for the adjustments incorporating the "observations" from the association models constructed with uniform distributions. Once again, the gross parameters of the associations are recovered nicely, despite the presence of two models (numbers 5 and 6) that seem to be statistical "flukes". These models exert a strong influence on the average absolute deviations, causing them to be higher than their Gaussian counterparts above. If the averages are formed without the values belonging to uniform associations 5 and 6, one obtains results almost identical to the numbers in the last line of table 4-3. Judging by the numbers in tables 4-1, 4-2, 4-3, and 4-4, it would appear that the adjustment with stochastic constraints is not terribly sensitive to the forms of the association distance and velocity distribution functions, provided that suitable constraints are chosen.

Table 4-3. Global parameters for uniform association models

Model No.	v_{xc} (km s^{-1})	v_{yc} (km s^{-1})	v_{zc} (km s^{-1})	π_c (arcsec.)
1	-1.91 ± 1.67	44.70 ± 1.06	6.30 ± 0.32	0.0193 ± 0.0020
2	-0.16 ± 1.40	43.94 ± 0.90	6.79 ± 0.26	0.0214 ± 0.0020
3	-2.30 ± 1.47	44.84 ± 0.94	6.36 ± 0.28	0.0191 ± 0.0018
4	-1.91 ± 1.67	44.70 ± 1.06	6.30 ± 0.32	0.0193 ± 0.0020
5	-4.55 ± 1.82	46.40 ± 1.16	5.99 ± 0.35	0.0172 ± 0.0018
6	0.99 ± 1.31	43.06 ± 0.81	6.74 ± 0.25	0.0224 ± 0.0020
7	-2.38 ± 1.51	45.25 ± 0.97	6.22 ± 0.30	0.0191 ± 0.0018
8	-0.75 ± 1.51	44.06 ± 0.96	6.48 ± 0.28	0.0209 ± 0.0020
9	-0.37 ± 1.43	43.77 ± 0.91	6.56 ± 0.27	0.0210 ± 0.0020
10	-0.50 ± 1.44	44.01 ± 0.92	6.52 ± 0.28	0.0209 ± 0.0020
Average	-1.38 ± 1.55	44.47 ± 0.93	6.43 ± 0.24	0.0201 ± 0.0015

Table 4-4. Deviations of stochastic parameters from true values for uniform models.

Model No.	$\lvert \Delta v_x \rvert$ (km s^{-1})	$\lvert \Delta v_y \rvert$ (km s^{-1})	$\lvert \Delta v_z \rvert$ (km s^{-1})	$\lvert \Delta\pi \rvert$ (mas)
1	0.50	0.24	0.22	0.9
2	1.25	0.57	0.30	1.3
3	0.89	0.34	0.16	0.9
4	0.50	0.24	0.22	0.9
5	3.14	1.92	0.50	2.8
6	2.40	1.47	0.26	2.4
7	0.97	0.73	0.28	1.1
8	0.66	0.46	0.13	1.0
9	1.04	0.76	0.16	1.1
10	0.91	0.51	0.14	1.1
Average	1.23	0.72	0.24	1.4

The adjustments involving the last group of four models with a different cluster center and velocity yielded no surprises, demonstrating that the adjustment technique is not sensitive to either the association's location or the direction of its average velocity. Therefore, for the sake of brevity, a summary of these models shall not be presented.

Comparison with Convergent Point Solutions

In chapter 2, much ado was made of the fact that an adjustment by means of least squares with probabilistic constraints should yield more accurate parameter estimates than other techniques, as it incorporates all kinematical observations and relaxes the unrealistic assumption of a shared velocity. The model associations with the Gaussian distributions presented a perfect opportunity to test this statement, and so some of the ten models in the first group were incorporated into two more adjustments. Parameters for models 1, 2, 4, 6 and 8 were first estimated by the stochastic constraint technique, with the parallax constraint tightened to a value of 0".002 (a 5 parsec dispersion). Then a solution was obtained for each of the models by means of a classical convergent point technique, similar to that used by Wayman et al. (1965). Finally, the estimates obtained by each of the methods were compared to the true values to determine their accuracy.

Table 4-5 is a summary of the stochastic constraint adjustments; note that the global velocity parameters are not significantly different from the corresponding values in table 4-1. The most important values for the current discussion are the average deviations from the true parallaxes, $\Delta\pi$, and the average absolute deviation, $|\Delta\pi|$. A glance at table 4-5 and table 4-6 is sufficient to show that the stochastic constraint adjustments yield more accurate parallaxes than those estimated by the convergent point method (43% in the case of $|\Delta\pi|$ and 100% with regard to $\Delta\pi$). A comparison of the individual stellar velocities is somewhat meaningless, as the convergent point solution yields only one velocity, shared by all stars in the association.

Table 4-5. Gaussian associations with parallax constraints of 2 mas.

Model No.	v_{xc} (km s^{-1})	v_{yc} (km s^{-1})	v_{zc} (km s^{-1})	π_c (mas)	$\Delta\pi$ (mas)	$\lvert\Delta\pi\rvert$ (mas)
1	-1.3 ± 1.2	44.6 ± 0.8	6.5 ± 0.2	199 ± 1.6	.03	.4
2	-0.5 ± 1.2	44.0 ± 0.8	6.5 ± 0.3	208 ± 1.7	.8	.9
4	-1.8 ± 1.2	44.9 ± 0.8	6.4 ± 0.2	195 ± 1.5	-.6	.7
6	-0.8 ± 1.1	44.2 ± 0.7	6.6 ± 0.2	204 ± 1.6	.4	.7
8	-0.9 ± 1.1	44.3 ± 0.7	6.4 ± 0.2	204 ± 1.6	.3	.6
Average	-1.1	44.4	6.5	202	.2 ± .5	.7 ± .2

Table 4-6. Classical convergent point solutions.

Model No.	α, δ CP	$\Delta\pi$ (mas)	$\lvert\Delta\pi\rvert$ (mas)
1	91°.7, 8°.2	.1	.8
2	90°.7, 8°.4	.9	1.2
4	91°.8, 8°.2	-.08	.8
6	91°.5, 8°.3	.2	.9
8	90°.8, 8°.4	.8	1.1
Average	91°.3, 8°.3	.4 ± .4	1.0 ± 0.2

More evidence of the superiority of the stochastic constraint technique is provided by figure 4-5, which shows the distributions of the true parallaxes, the parallaxes obtained by the stochastic adjustment, and the convergent point parallaxes for one of the model associations. The stochastic parallax distribution (figure 4-5b) is a fair approximation to that of the true parallaxes (figure 4-5a), but the convergent point parallax distribution (figure 4-5c) is much wider than that of the true values. The stochastic adjustment has preserved the shape of the association, whereas the convergent point parallaxes, as expected, "smear out" the stars, distorting the association along the line of sight. Tables 4-5 and 4-6, combined with figure 4-5, should provide sufficient reason for anyone concerned with accuracy to abandon the determination of parallaxes by means of the classical convergent point approach.

However, it would be misleading to state that parameters obtained by an adjustment with stochastic constraints will always be accurate estimators of the true values. As with

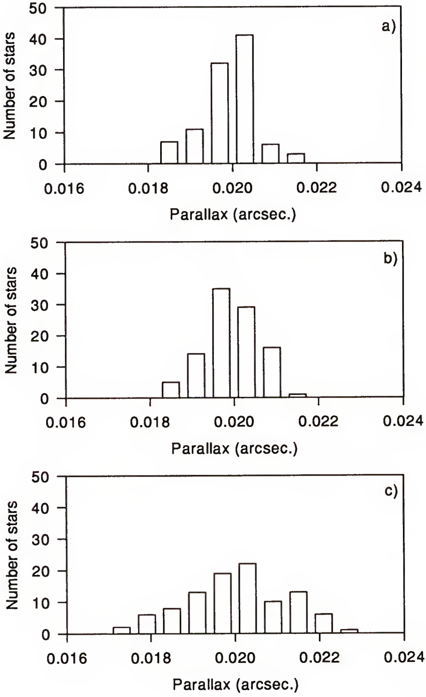

Figure 4-5. Distribution of model parallaxes.
a) "True" parallaxes of model stars; b) Parallaxes of model stars obtained by stochastic constraint technique; c) Parallaxes of model stars obtained by classical convergent point solution.

any least-squares procedure, the technique assumes that the observations are bias-free; use of biased observations will propagate the bias into the parameter estimates. Figure 4-6 illustrates the effect of such a bias upon the solution. The parallax estimates obtained by the adjustment (figure 4-6c) again reproduce the form of the true distribution (figure 4-6a), but are offset by approximately -0".001. As all the "observed" parallaxes of the model stars enter into the adjustment with equal weight (being from the same distribution), it is thought that the near-zero parallaxes are responsible for the shift of the association's center from 50 parsecs to about 53 parsecs. The fault lies not with the specific technique of probabilistic constraints; rather, it lies with the inability of least squares to handle biased data.[3] The lesson here is clear; use of biased observations in a least-squares adjustment should be avoided if at all possible, unless the bias can be removed by some means. This will have an important bearing on the choice of the data to be used in the determination of the Hyades distance.

Application of the Technique to the Hyades

Validity of Assumptions and Determination of Constraints

In the adjustment procedure we will stochastically constrain the velocity components and the parallaxes of the Hyades stars to obey Gaussian distributions. It has been shown previously in this chapter that the parameter estimates obtained by the least squares with probabilistic constraints are not terribly sensitive to the forms of these distributions; still, it would be reassuring if some idea of the actual distributions could be obtained before performing the adjustment. Fortunately, there exist excellent dynamical studies of the

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[3] This bias originates in the fact that randomly distributed small parallaxes introduce a bias into the distances because $dr = -\frac{d\pi}{\pi^2}$; other things being equal, a change in a small parallax produces a larger change in the corresponding distance than the same change in a large parallax.

Hyades by van Bueren (1952), Oort and Pels (1975), Gunn et al. (1988) and Reid (1992) that shed considerable light on the nature of the distance and velocity distributions.

Oort and Pels (1975) found that the surface density of the Hyades stars drops off exponentially with distance from the center of the cluster. Figure 4-7 is a comparison of their surface density distribution with that of Reid (1992), who, along with Gunn et al. (1988), found that the surface density distribution of the Hyades stars could be characterized by a Plummer model. In this simple polytropic approximation, the surface density of the stars is given by

$$\Sigma(r) = N\left(1+\frac{r^2}{A^2}\right)^{-2}, \tag{4-11}$$

where r is the angular distance in degrees from the center of the cluster and N and A are constants. Note that in the case of the Hyades, the Plummer model and the exponential model of Oort and Pels give approximately the same results, which is not too surprising as the cluster stars show a concentration towards the center. It should be pointed out that the distributions shown in figure 4-7 are for the brighter members of the Hyades, as the Hyades dwarfs are more dispersed and have a lesser affinity for the center. As a result, Oort and Pels give two exponential spatial density distributions, one for the dwarfs and the other for the brighter stars (mainly of spectral type A). Figure 4-8 shows their observed number distribution function for the brighter Hyads. It should now be obvious that the distance distribution of the Hyades stars is anything but uniform, and that a Gaussian with a suitably chosen width can approximate this distribution.

An approximate width of the distance distribution can be determined from figure 4-8, which would seem to indicate a value of about 6 parsecs. This is borne out by an examination of figure 4-9, which is a plot of the equatorial positions of the Hyades stars with visual magnitudes brighter than 9.5. Even though the cluster is of large extent

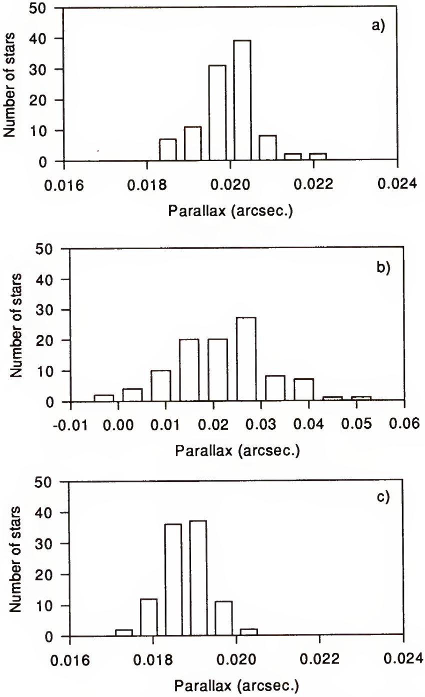

Figure 4-6. Model solution showing bias in stochastic parallaxes.
a) True parallaxes of model stars; b) "Observed" parallaxes; c) Stochastic parallaxes.

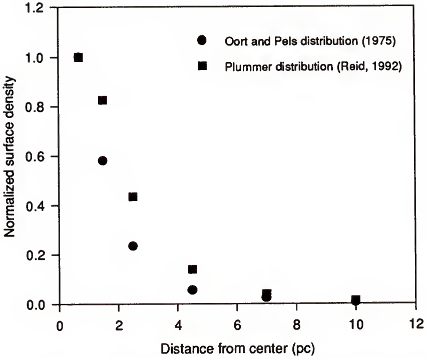

Figure 4-7. Normalized surface density distribution for Hyades stars.

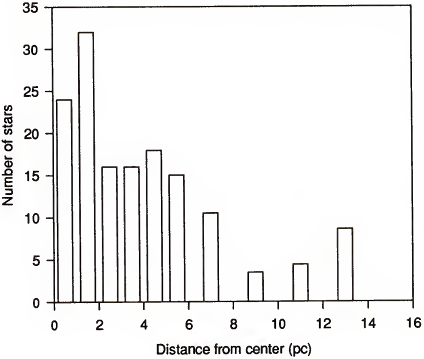

Figure 4-8. Observed distance distribution for the brighter Hyads.

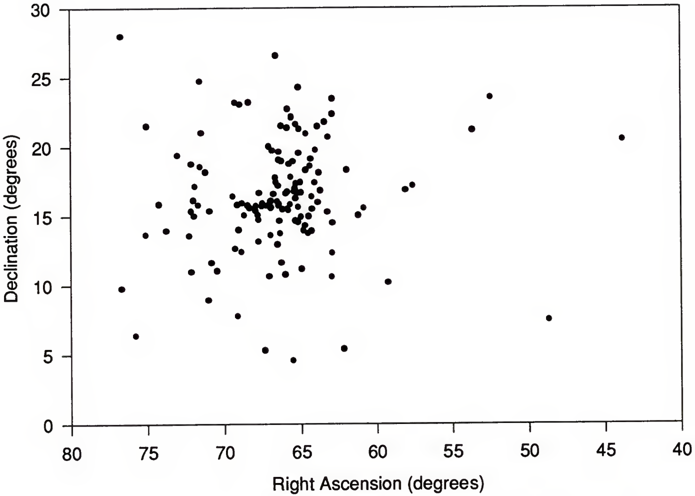

Figure 4-9. Equatorial positions of Hyades stars with $m_v > 9$ (orientation B1950.0).

(>15°), there is an obvious concentration of stars in the area defined by $63° < \alpha < 73°$, $13° < \delta < 23°$, or for a radius of about 5° from the center of the cluster. Assuming a Hyades distance of 48 parsecs, 5° corresponds to a distance of 4.2 parsecs from the center, in rough agreement with the value determined from figure 4-8. Given these values, and also recalling that the faint members of the cluster show less of a concentration towards the center, 6 parsecs as our distance dispersion appears to be a reasonable choice, resulting in a parallax constraint of approximately 0".003. It should be pointed out that even if we adopt the Hyades tidal limit of approximately 10 parsecs (see Oort and Pels, 1975 or Reid, 1992) as the dispersion width, the parallax constraint would increase by 0".002, to a value of 0".005.

Obtaining the dispersion of the stars' velocity components is much easier, thanks to the radial velocity studies. In their study of the Hyades, Oort and Pels (1975) assumed that the cluster is in dynamical equilibrium, with the internal motions of the member stars obeying a (truncated) Maxwellian velocity distribution. Using radial velocities from the Wilson catalogue, they estimated a velocity dispersion of approximately 0.25 km s^{-1} in each cartesian direction. Gunn et al. (1988) assumed a velocity distribution dictated by the adoption of the Plummer model and derived a dispersion value of 0.23 km s^{-1}, which they estimate cannot be in error by more than 20%. It is hard to argue with such good agreement; therefore, we shall adopt 0.25 km s^{-1} as the stochastic constraint for the velocities.

Choice of Data

The model associations, while demonstrating the superiority of the technique of least squares with probabilistic constraints over the conventional techniques, also illustrate the dangers of using biased observations in the adjustment. Biases are easy to discover in the virtual reality of computer models, but nature restricts us to obtaining only estimates of the quantities we wish to observe; the "true" values are forever out of reach, cloaked by the unavoidable errors in the observing process and the equally-unavoidable imperfections of the mathematical reduction models. It is hoped, and all too often assumed, that these errors are random in nature, and that as the precision of the observations increases, so to does their accuracy. Unfortunately, the history of astrometry is replete with examples of estimates based on inaccurate, but precise, observations. One such example involves Hyad proper motion determinations made by various investigators; an examination of each investigators' statements concerning the accuracies of their motions relative to those of their colleagues evokes the feeling that one has just chanced upon a raging battle. Indeed, the fury of the criticisms (most of which have at least a degree of validity) tends to cast doubt as whether there exist any proper motions reasonably free of systematic effects.[4]

Nonetheless, proper motions are a necessary element in determining the Hyades distance. They must be incorporated into the adjustment, requiring the development of some sort of selection procedure. One possible approach is to combine all the available Hyades proper motions into one huge data set and hope that the systematic errors will "cancel each other out". However, these proper motions are a bewildering mess of differing measurement techniques and coordinate systems, thus rendering such an adjustment impracticable, if not impossible. Another approach would be to adopt the concept of "newer is better" and consider only the most modern proper motions in the adjustment, justified, perhaps to some extent by the consideration that proper motions with respect to remote galaxies are probably less affected by systematic errors than those based on transit circle observations and other differential photographic techniques. This approach is followed by most investigators, and while the improvements in instruments and advances in data acquisition give it some credence, it does a disservice to the older proper motion estimates. Even more importantly, the information provided by these older motions is ignored. It would seem that there is no good approach, the choice being between loss of information by ignoring past data or reliance on the chance cancellation of the systematic errors of the various proper motions in a global adjustment.

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[4] Eichhorn (1993, private communication) relates that at the First Cosmic Distance Conference, held in 1955 at the University of Virginia in Charlottesville, W. J. Luyten began an invited talk saying: "I am supposed to talk about proper motions. Unfortunately, there exist only improper motions. . .".

However, it is possible to compare the proper motions of stars common to multiple investigations; hopefully, such a comparison will provide some insight as to which proper motions suffer from bias the least. It has long been known that the transit circle proper motions used in the older Hyades distance determinations suffer from regional systematic errors that limit their usefulness (Hanson, 1975; Hanson, 1980; Reid, 1992). The sole exception seems to be the FK 5 proper motions determined by Schwan (1992), which yield a Hyades distance compatible with that obtained by the photographic proper motions (Reid, 1992). This would indeed be a fortunate circumstance, as the radial velocities of 140 of his 145 Hyades members have been determined as well as the absolute parallaxes of 37 of these stars (explained by the fact that the Schwan stars have visual magnitudes brighter than 9.5).

Systematic errors plague the galaxy-referred photographic proper motions as well. Hanson's (1975) proper motions were touted as being free of systematic effects due to the use of external galaxies as reference objects. Two years later, McAlister (1977) showed that these proper motions suffered from a magnitude dependency, and a comparison with Reid (1992) also shows a significant discrepancy between his estimates and those of Hanson. However, the Schmidt plate proper motions of Luyten et al. (1981) and Reid (1992) agree nicely and have not been shown to suffer from major systematic effects as of this date, making them good candidates for incorporation into an adjustment. Unfortunately, the quoted precision of these motions is much less than that of Hanson's (1975) or even that of Schwan's (1991); it would seem that accuracy has been improved at the cost of precision.

The use of comparisons seems to reduce the set of usable proper motion observations to those of Schwan (1991), Luyten et al. (1982) and Reid (1992). The older proper motions of Smart (1939), Van Bueren (1952), Corbin et al. (1975) and Oort and Pels (1975) are not good candidates for use in adjustments, not because they are older and more imprecise, but because they suffer from serious systematic effects that would bias the parameter estimates. Hanson's (1975) proper motions could possibly be corrected for the magnitude effect, but the rewards would probably be small, as there is a large degree of overlap between his data and that of Reid (1992).

There are not many trigonometric parallaxes available, and those that exist for Hyades members are incorporated into the adjustment, provided they were reduced to an absolute system (van Altena, 1974a). To a lesser extent this is also true of the radial velocities, where the observations come from three sources: Wilson's catalogue (standard error of ~5 km s^{-1}), the radial velocities of Detweiler et al (1984) (standard error of ~0.25 km s^{-1}), and the radial velocities of Griffin et al. (1988) (standard error of ~0.25 km s^{-1}). There may be a zero point error (bias) of about 1 km s^{-1} in the high precision estimates due to the difficulties involved in the calibration procedure. However, this is not certain, making these velocities prefered over those from Wilson's catalogue due to their much higher precision.

Table 4-7 lists the observations for the 140 stars chosen for the adjustment. The first column lists the Schwan (1992) number, and the next two columns give the star's B1950.0 position (in units of decimal degrees). Columns 4 through 7 list the proper motion components and their associated standard errors (units are 10^{-4} arcseconds). The radial velocity data are contained in columns 8 and 9; a G in column 9 indicates that the velocity is from Gunn's (1988) data, whereas a W indicates that value is from Wilson's catalogue. No errors are given for the radial velocities; these are assumed to be 0.25 km s^{-1} for Gunn's estimates and 5 km s^{-1} for Wilson's values. Columns 10 and 11 list the visual magnitudes and the B-V indices for the stars. Finally, the observed parallax (if available) and its standard error are given for each star in columns 12 and 13 (parallax units are ", whereas the error is given in units of mas).

The Adjustment.

Once the constraints had been established and the observations selected, an adjustment was performed using a FORTRAN program similar to that given in appendix A. The starting values of the parameters were set at

$$\begin{aligned} \dot{x}_0 &= -5 \\ \dot{y}_0 &= 45 \\ \dot{z}_0 &= 5.5 \\ \pi_0 &= 0''.0225 \end{aligned}$$

Convergence was achieved after ten iterations, and the resulting global parameters are presented in table 4-8. Note that the parallax estimate places the cluster center at 49.8 ± 1.1 parsecs, a value in good agreement with recent determinations (Schwan, 1991 and Reid, 1992). The residuals are plotted in figures 4-11 through 4-15 (they are listed in appendix B). Note that there are no obvious systematic errors in any of the residuals with respect to position or the stars' visual magnitude.

The stochastic parameters, the "gems in the crown" of this adjustment procedure, are listed in table 4-9. The velocity components have standard deviations of 0.22, 0.31 and 0.40 km s^{-1} in the equatorial x, y, and z directions, respectively. The stochastic parallaxes have a standard deviation of 0".0019 (4.7 parsecs) about their mean of 0".0201, with extrema of 0".0301 (33.2 parsecs) and 0".0151 (66.2 parsecs). These parallaxes are depicted in figure 4-10.

Table 4-7. Observations chosen for incorporation into the adjustment.

No.	α	δ	μ_α	ε_α	μ_δ	ε_δ	r_v	I	m_v	B-V	π	ε_π
1	64.2375	15.5167	1198	4	-245	4	39.30	G	3.64	0.99	0.0263	100
2	65.0125	17.4333	1122	4	-291	5	39.70	G	3.76	0.98	0.0188	90
3	66.4250	19.0667	1135	3	-372	4	39.40	G	3.54	1.02	0.0155	131
4	75.0250	21.5167	706	4	-414	5	42.20	W	4.64	0.15	0.0185	44
5	43.8042	20.4667	2488	6	-298	6	28.50	W	5.81	0.41		
6	57.5750	17.1833	1485	7	-298	7	35.00	W	5.97	0.34	0.0310	85
7	67.7500	14.7333	1063	6	-261	7	37.50	W	4.65	0.25	0.0241	85
8	72.1125	18.7500	843	6	-345	7	38.50	W	5.10	0.22	0.0214	80
9	76.6542	27.9667	619	9	-598	9	45.00	G	5.99	0.27		
10	76.6417	9.7667	645	7	-71	8	37.20	W	5.42	0.24	0.0099	119
11	62.1667	5.4000	1492	10	55	11	36.60	W	5.72	0.36	0.0320	92
12	63.8542	21.4667	1014	9	-366	10	35.40	W	5.65	0.28		
13	65.8292	22.7000	1162	13	-458	13	35.10	W	4.28	0.26	0.0348	170
14	66.5083	12.9333	1074	19	-156	22	33.40	W	5.03	0.22		
15	68.8417	12.4167	1057	16	-128	19	45.00	W	4.27	0.12	0.0216	72
16	69.1000	7.7667	948	19	-29	22	36.30	W	5.39	0.26		
17	69.1042	15.8167	864	13	-181	14	36.90	W	4.69	0.15		
18	70.8125	11.6167	691	19	-52	23	40.80	W	5.37	0.20	0.0043	85
19	59.2333	10.1833	1419	33	-62	30	39.60	W	6.38	0.41		
20	63.2333	15.2833	1174	22	-245	21	36.90	W	6.33	0.40		
21	64.2875	13.9167	1216	22	-182	24	42.00	W	5.59	0.28		
22	64.4417	14.9833	1131	18	-202	22	36.20	W	5.26	0.23		
23	64.5167	13.7500	1092	18	-207	17	37.00	W	6.18	0.46		
24	64.8083	13.9667	1143	16	-240	19	41.20	W	5.72	0.31		
25	64.9667	16.6833	1464	39	-275	32	33.00	W	7.86	0.68		
26	65.1375	16.6667	1123	18	-174	22	35.00	W	5.64	0.30		
27	65.3042	17.3333	1134	13	-318	16	37.50	W	4.80	0.16	0.0167	136
28	65.5083	18.9333	1125	16	-345	16	36.60	W	5.98	0.37		
29	65.5958	22.1833	1135	19	-407	19	40.10	W	4.22	0.14	0.0229	138
30	65.6083	22.0833	1197	19	-442	19	32.00	W	5.27	0.24		
31	65.6500	17.8167	1158	13	-364	17	40.20	G	4.29	0.05	0.0232	28
32	65.6917	15.8333	1050	31	-253	35	36.40	W	6.46	0.49	0.0370	153
33	66.2583	21.5167	1059	16	-413	18	36.20	W	5.72	0.27		
34	66.4292	15.8500	1096	15	-293	15	40.20	G	3.85	0.95	0.0369	155
35	66.4500	15.7667	1117	16	-255	18	39.50	W	3.40	0.18	0.0272	170
36	66.9250	16.0833	1092	16	-239	19	37.50	W	4.78	0.17	0.0627	101
37	66.9500	15.5833	1072	12	-247	18	39.30	W	5.48	0.26		
38	66.9500	13.6167	1135	21	-193	25	38.80	W	5.40	0.26		
39	67.2500	15.7500	1075	24	-295	23	36.00	W	6.02	0.34		
40	68.6708	15.0500	1095	19	-292	18	42.00	G	7.54	0.61		
41	68.8208	15.9333	934	16	-286	19	38.40	W	5.78	0.31		
42	68.9625	23.0500	1113	22	-578	19	39.90	G	7.53	0.58		
43	70.4125	11.0500	984	21	-119	26	39.40	W	5.40	0.25		
44	74.2167	15.8500	876	15	-322	16	44.10	W	6.75	0.45		
45	52.4708	23.5167	1755	39	-353	37	32.70	G	8.89	0.84		
46	53.6667	21.1833	1635	36	-354	33	33.60	G	9.37	0.92		
47	62.8833	22.3333	1035	21	-265	20	38.40	W	7.05	0.42		
48	62.8833	23.4500	1351	36	-514	33	38.20	G	8.08	0.66		
49	63.1917	20.7000	1216	21	-355	21	38.50	G	7.84	0.71		

Table 4-7 -- continued

No.	α	δ	μ_α	ε_α	μ_δ	ε_δ	r_v	I	m_v	B-V	π	ε_π
50	63.3958	21.7833	1171	21	-425	19	38.50	G	9.15	0.82		
51	64.6417	20.9167	1126	21	-365	20	37.10	W	6.80	0.44		
52	65.1042	21.2667	1081	39	-405	38	39.20	G	8.85	0.76		
53	65.1208	24.2833	1021	24	-455	22	34.10	W	7.18	0.45		
54	65.3167	21.6167	1276	21	-484	21	38.90	G	7.13	0.52		
55	65.8750	21.3667	1111	21	-345	20	38.80	G	7.38	0.54		
56	66.6083	26.5667	991	27	-516	23	39.00	G	9.19	0.76		
57	67.0792	20.0333	1066	39	-345	38	40.60	G	8.59	0.74		
58	68.3667	23.2333	1141	25	-495	24	43.30	W	6.05	0.39		
59	69.2708	23.2000	721	40	-327	37	41.30	G	9.40	0.81		
60	71.4333	21.0167	916	19	-436	18	41.80	G	9.07	0.84		
61	71.5417	24.7167	993	43	-555	39	40.60	G	9.59	1.05		
62	76.6542	27.9667	543	51	-477	42	45.00	G	8.59	0.69		
63	58.0625	16.8500	1500	22	-274	22	36.80	G	8.99	0.90		
64	60.8583	15.5667	1260	42	-215	42	38.00	G	7.85	0.59		
65	61.2167	15.0333	1440	51	-254	44	36.40	G	6.02	0.40		
66	61.9542	18.3000	1290	24	-264	23	33.00	W	6.62	0.42		
67	62.9000	14.5000	1215	42	-235	38	39.20	G	8.46	0.70		
68	62.9167	12.3167	1185	24	-55	22	39.30	G	8.06	0.64		
69	62.9542	10.5833	1200	24	-155	23	38.50	G	7.12	0.51		
70	63.6958	16.8333	1125	22	-265	21	38.20	G	8.32	0.76		
71	63.7833	18.1333	1170	19	-295	20	44.00	W	7.53	0.68		
72	63.8667	15.9667	1245	22	-275	21	39.60	G	9.59	0.98		
73	64.0083	19.7833	1186	39	-395	38	39.60	G	8.63	0.74		
74	64.0625	17.4000	1155	22	-205	22	38.80	G	8.43	0.73		
75	64.2625	16.4000	1185	48	-285	41	39.20	G	6.89	0.56		
76	64.3250	19.1167	1231	21	-355	21	38.80	G	7.47	0.57		
77	64.3792	18.6167	1215	25	-305	24	42.00	W	6.13	0.37	0.0190	58
78	64.6583	18.3000	1170	22	-235	21	37.00	W	6.81	0.44		
79	64.6875	14.3000	1125	25	-155	24	39.20	G	6.61	0.41		
80	64.9167	11.1833	1215	22	-155	22	40.80	G	9.81	1.18		
81	64.9750	14.9333	1320	45	-184	40	42.70	G	6.99	0.56		
82	65.1125	19.5500	1036	39	-405	39	39.90	G	9.40	0.91		
83	65.1417	15.6500	1276	55	-364	54	39.30	G	10.50	1.23	0.0304	90
84	65.1750	14.5500	1215	22	-185	21	40.30	G	9.11	0.86		
85	65.3417	14.6500	1185	22	-205	22	37.50	G	7.62	0.60	0.0197	36
86	65.3375	16.2667	1035	42	-245	41	39.80	G	8.24	0.59		
87	65.3750	16.9667	1051	46	-285	40	43.20	W	6.97	0.44		
88	65.4000	16.7667	1260	42	-254	41	38.80	G	7.80	0.60		
89	65.7375	18.7500	1201	36	-315	35	40.70	G	7.53	0.68		
90	65.8125	15.4167	1111	24	-285	22	40.00	G	7.48	0.54		
91	65.8750	15.5000	1170	48	-235	47	41.40	W	4.49	0.25	0.0221	49
92	65.8833	16.7333	1170	25	-245	24	39.40	G	8.03	0.65	0.0317	74
93	65.9500	16.6333	1155	24	-285	24	40.20	G	8.10	0.67	0.0200	65
94	66.0125	10.7667	1065	24	-115	22	41.10	G	9.46	1.03		
95	66.1875	15.4833	1140	25	-265	24	39.30	G	7.42	0.54	0.0321	137
96	66.2458	11.6333	1080	24	-105	22	40.20	G	7.51	0.55		
97	66.2583	18.9500	856	27	-316	26	41.00	G	11.00	1.20		
98	66.3125	13.7500	1005	48	-126	45	40.90	G	9.02	0.84		
99	66.3875	14.6333	1095	28	-195	26	44.20	W	5.89	0.32		

Table 4-7 -- continued

No.	α	δ	μ_α	ϵ_α	μ_δ	ϵ_δ	r_v	I	m_v	B-V	π	ϵ_π
100	66.4208	19.6333	1051	37	-345	38	39.90	G	8.63	0.75	0.0254	31
101	66.4792	17.1833	1171	22	-315	21	40.00	G	7.85	0.60	0.0458	200
102	66.5333	16.0500	1020	43	-225	38	45.00	W	6.59	0.53		
103	66.6125	17.4333	1111	24	-355	22	39.90	G	7.04	0.50		
104	66.6542	17.7500	1111	28	-355	29	38.60	G	6.91	0.45	0.0309	100
105	66.6583	17.7833	1126	21	-325	20	40.70	G	8.96	0.83		
106	66.7708	16.5667	1066	22	-275	21	41.00	G	9.10	0.86		
107	66.8417	19.7333	1021	19	-325	20	38.00	W	7.10	0.47		
108	66.9333	15.6333	975	24	-236	22	40.80	G	8.94	0.85	0.0214	29
109	66.9792	16.0500	1066	27	-275	26	43.60	W	6.51	0.43		
110	67.0458	10.6500	930	25	-116	23	41.40	G	7.04	0.47	0.0171	24
111	67.4917	15.9000	931	42	-346	41	43.00	W	8.94	0.88	0.0465	99
112	67.5333	15.7167	946	22	-266	21	41.60	G	8.65	0.74		
113	67.6875	16.6667	841	42	-296	40	41.30	G	9.41	0.89	0.0326	48
114	67.7417	13.1500	975	21	-116	21	35.00	W	6.62	0.43	0.0345	46
115	67.7833	15.0667	991	19	-296	17	37.60	W	8.49	0.84		
116	67.9208	15.7167	811	24	-236	22	41.10	G	9.67	0.92		
117	67.9333	15.4000	961	22	-306	22	40.80	G	7.89	0.64		
118	68.3083	15.5833	856	42	-326	41	41.50	G	9.38	0.86	0.0184	33
119	68.4542	15.7667	901	27	-306	25	33.70	W	6.65	0.44		
120	69.0333	14.0000	990	22	-235	21	41.90	G	7.96	0.66		
121	69.2625	12.6333	1020	24	-155	23	42.50	G	10.04	1.08		
122	69.3875	16.4167	916	49	-336	46	40.90	G	9.47	1.10		
123	70.9125	15.3833	886	22	-326	21	42.40	G	8.32	0.67		
124	71.1792	18.1667	916	22	-406	22	42.00	W	8.53	0.72		
125	71.5333	18.5500	961	21	-426	20	41.60	G	8.99	0.83		
126	71.6667	15.8000	886	22	-286	21	42.00	G	7.74	0.58		
127	71.8750	17.1167	1006	24	-365	24	41.70	W	7.11	0.56		
128	71.9292	15.0000	841	22	-226	21	42.30	G	7.59	0.73		
129	71.9833	16.1333	826	27	-336	25	42.70	G	7.29	0.50		
130	72.1083	10.9833	825	48	-136	42	42.40	G	6.77	0.53		
131	72.1333	15.3500	646	24	-207	23	42.90	G	7.89	0.53		
132	72.2542	13.5667	795	48	-206	44	38.80	W	6.27	0.50		
133	73.0083	19.4000	721	42	-337	39	35.00	W	6.37	0.29		
134	73.7500	13.9333	660	21	-187	19	43.70	G	8.89	0.74	0.0247	29
135	75.0750	13.6667	645	48	-137	44	43.50	G	9.00	0.76		
136	48.6875	7.4667	1725	22	-54	21	32.40	G	7.39	0.57	0.0267	37
137	65.5167	4.5833	1020	40	55	38	42.20	G	8.93	0.76		
138	67.3542	5.3000	1140	25	115	24	40.30	W	6.38	0.41		
139	71.0083	8.9333	1080	25	-45	23	42.50	G	7.26	0.56		
140	75.7458	6.4000	675	48	-37	47	42.90	G	9.92	0.95		

Table 4-8. Hyades cluster parameter estimates.

$\dot{x}_c$ (km s^{-1})	$\dot{y}_c$ (km s^{-1})	$\dot{z}_c$ (km s^{-1})	π_c (mas)
-7.50 ± 0.41	47.11 ± 0.23	5.23 ± 0.13	20.1 ± 0.4

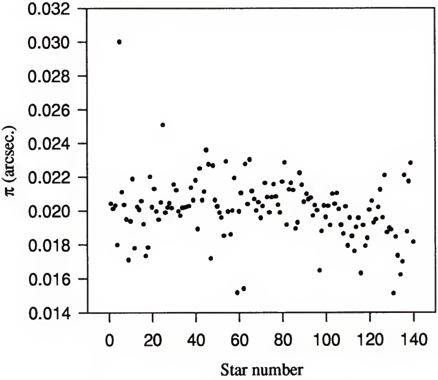

Figure 4-10. Stochastic parallax estimates.

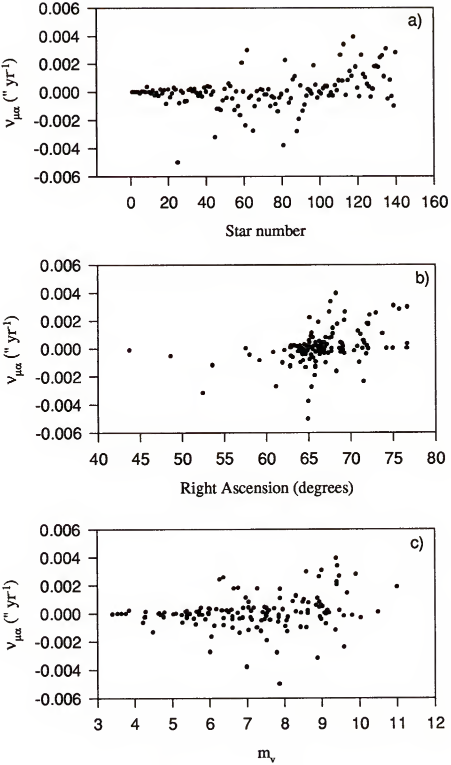

Figure 4-11. Residuals in $\mu\alpha$.
a) Residuals in μ_α; b) Residuals in μ_α vs. α; c) Residual in μ_α vs. m_v.

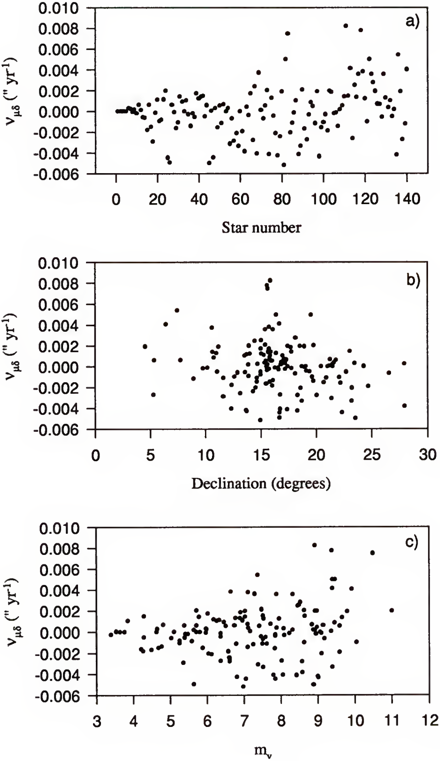

Figure 4-12. Residuals in μ_δ.
a) Residuals in μ_δ; b) Residuals in μ_δ vs. δ; c) Residuals in μ_δ vs. m_v.

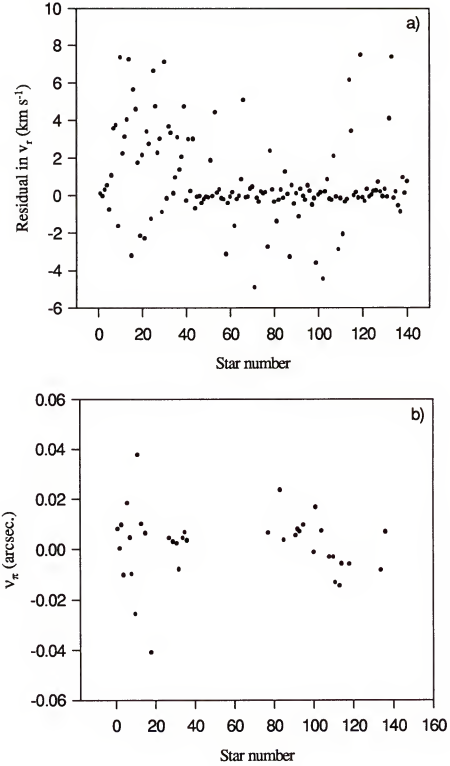

Figure 4-13. Residuals in the radial velocity and parallax. a) Residuals in the radial velocity; b) Residuals in π.

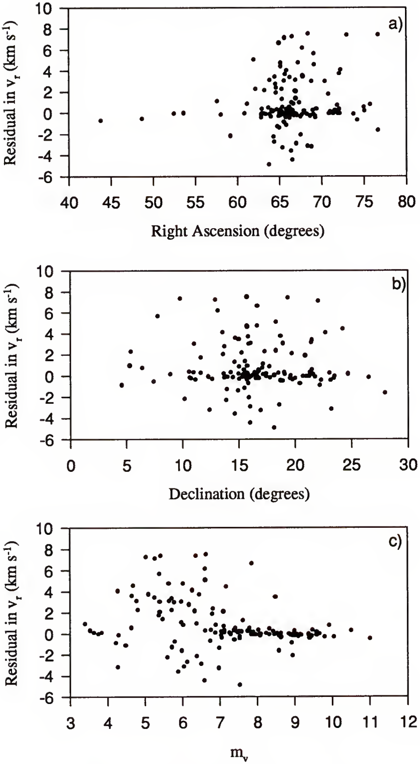

Figure 4-14. Residuals in the radial velocity.
a) Residuals in v_r vs. α; b) Residuals in v_r vs. δ; c) Residuals in v_r vs. m_v.

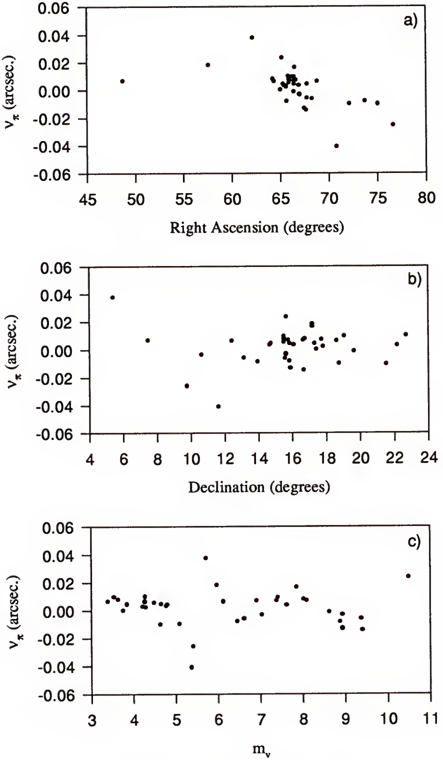

Figure 4-15. Residuals in the parallax.
a) Residuals in π vs. α; b) Residuals in π vs. δ; c) Residuals in π vs. m_v.

Table 4-9. Stochastic velocities and parallaxes for 140 Hyades members.

No.	$\dot{x}$ (km s^{-1})	$\dot{y}$ (km s^{-1})	$\dot{z}$ (km s^{-1})	π (arcsec.)
1	-7.1070	47.0220	5.2040	0.0204
2	-7.0690	47.1170	5.3160	0.0201
3	-7.1580	46.8440	5.0770	0.0203
4	-6.9650	47.0570	5.3010	0.0180
5	-7.2800	47.3410	5.2840	0.0300
6	-6.8930	47.2210	4.6350	0.0211
7	-6.9040	47.1710	4.6820	0.0203
8	-7.0630	47.0540	5.2970	0.0195
9	-6.2250	48.5490	5.7550	0.0171
10	-7.0620	46.9910	5.5090	0.0194
11	-7.1300	47.1900	4.6930	0.0219
12	-6.8860	47.0690	5.0680	0.0178
13	-7.2470	47.0210	5.5250	0.0202
14	-7.1030	47.0290	5.3430	0.0201
15	-7.1800	47.0760	5.7590	0.0206
16	-7.0130	47.0600	5.0920	0.0192
17	-7.2090	46.8900	6.0050	0.0173
18	-7.0530	46.9740	5.7980	0.0178
19	-7.1060	47.1710	5.2510	0.0220
20	-6.9840	47.1370	4.9720	0.0202
21	-7.1290	47.1330	5.4010	0.0213
22	-7.1060	47.0540	5.3970	0.0200
23	-6.9270	47.1370	4.8170	0.0195
24	-6.9080	47.2190	4.6650	0.0205
25	-7.4470	47.0830	5.6580	0.0251
26	-7.4280	46.8970	6.2590	0.0199
27	-6.9620	47.1380	4.9700	0.0202
28	-7.0860	47.0890	5.2510	0.0204
29	-7.2940	47.0350	5.6820	0.0201
30	-7.2970	47.0270	5.4910	0.0216
31	-6.8330	47.3560	4.7730	0.0212
32	-7.0290	47.0910	5.1450	0.0200
33	-7.0430	47.0760	5.2240	0.0197
34	-6.9360	47.0790	4.7180	0.0202
35	-7.0840	47.0920	5.3190	0.0202
36	-7.2020	47.0240	5.6160	0.0202
37	-7.0810	47.0940	5.2930	0.0203
38	-7.1360	47.1060	5.3170	0.0214
39	-6.9750	47.1350	4.8860	0.0206
40	-6.9090	47.4490	4.9100	0.0218
41	-6.8510	47.1310	4.8180	0.0189
42	-6.9850	47.0380	4.7340	0.0225
43	-7.1000	47.0840	5.3060	0.0206
44	-7.0070	47.1790	4.9970	0.0211
45	-7.2820	47.1730	5.6150	0.0236
46	-7.1100	47.1920	5.2360	0.0227
47	-7.5070	46.8490	6.3270	0.0172
48	-7.0500	47.3390	5.2880	0.0227
49	-7.2150	47.1020	5.6030	0.0206

Table 4-9 -- continued

No.	$\dot{x}$ (km s^{-1})	$\dot{y}$ (km s^{-1})	$\dot{z}$ (km s^{-1})	π (arcsec.)
50	-6.9320	47.2230	5.1080	0.0202
51	-7.1840	47.0410	5.5050	0.0199
52	-6.9910	47.1330	5.1960	0.0196
53	-6.9930	47.0450	5.2110	0.0185
54	-7.1850	47.0660	5.1060	0.0229
55	-7.5760	46.6830	5.8860	0.0199
56	-7.0120	47.1800	5.4170	0.0186
57	-7.0620	47.2580	5.4990	0.0200
58	-7.3530	47.1030	5.5990	0.0219
59	-6.8190	47.3280	5.6360	0.0152
60	-7.0240	47.1640	5.2620	0.0199
61	-7.2440	46.9500	5.2800	0.0210
62	-6.6170	48.3940	6.2090	0.0154
63	-7.0060	47.3520	5.2000	0.0228
64	-7.0450	47.1480	5.2630	0.0204
65	-7.5380	46.4500	4.9160	0.0230
66	-7.3170	47.0020	5.7510	0.0212
67	-6.9780	47.2390	5.0840	0.0207
68	-7.1880	47.0520	6.1200	0.0200
69	-7.1200	46.8920	4.4260	0.0205
70	-7.2130	46.7040	5.0620	0.0196
71	-7.1030	47.1290	5.4240	0.0203
72	-7.0070	47.2900	5.1420	0.0216
73	-6.8680	47.4470	5.2100	0.0208
74	-7.4530	46.7750	6.0300	0.0199
75	-7.1010	47.0340	5.1200	0.0208
76	-7.2200	46.9810	5.2750	0.0215
77	-7.2210	47.0880	5.6100	0.0208
78	-7.4650	46.9300	6.2000	0.0203
79	-7.2890	46.7640	5.5880	0.0199
80	-6.9060	47.4960	4.8830	0.0217
81	-6.6940	48.3380	5.9250	0.0228
82	-6.7610	47.3580	4.9830	0.0192
83	-7.1370	46.9090	4.8860	0.0213
84	-7.1950	47.1880	5.7410	0.0216
85	-7.6890	45.9570	5.1420	0.0212
86	-7.0270	47.0280	5.2210	0.0189
87	-6.9730	47.1370	5.1740	0.0193
88	-7.4510	46.6310	5.3210	0.0222
89	-7.0550	47.4790	5.6520	0.0215
90	-6.9680	47.0900	4.7530	0.0205
91	-7.1070	47.1290	5.3130	0.0210
92	-7.3910	46.7280	5.6130	0.0207
93	-7.0900	47.1320	5.2950	0.0207
94	-6.9090	47.3330	5.0930	0.0197
95	-7.2370	46.6520	4.9930	0.0203
96	-7.1900	46.8490	5.4260	0.0200
97	-6.5630	47.5260	5.1070	0.0165
98	-6.9880	47.1880	5.4900	0.0188
99	-7.1090	47.1130	5.4730	0.0203

Table 4-9 -- continued

No.	$\dot{x}$ (km s^{-1})	$\dot{y}$ (km s^{-1})	$\dot{z}$ (km s^{-1})	π (arcsec.)
100	-7.0710	47.0620	5.2920	0.0196
101	-7.1410	46.9630	5.2490	0.0203
102	-7.0310	47.1130	5.3750	0.0191
103	-7.0070	47.0040	4.7500	0.0210
104	-7.2950	46.4010	4.7470	0.0204
105	-7.0380	47.2870	5.3360	0.0210
106	-6.9670	47.3180	5.3340	0.0201
107	-7.1840	47.0110	5.5840	0.0191
108	-6.9450	47.1380	5.2080	0.0186
109	-7.0120	47.1560	5.1560	0.0202
110	-6.8430	47.2430	4.9810	0.0179
111	-6.8380	47.2030	4.7370	0.0196
112	-6.7310	47.4340	4.9810	0.0185
113	-6.7540	47.2900	5.0290	0.0176
114	-7.1700	46.9460	5.8210	0.0190
115	-6.7480	47.2130	4.3360	0.0196
116	-6.7640	47.0890	4.9460	0.0163
117	-6.8060	47.0680	4.4330	0.0191
118	-6.7350	47.2530	4.7910	0.0179
119	-6.7690	47.1150	4.5880	0.0184
120	-6.9280	47.2430	4.9730	0.0200
121	-7.0160	47.3630	5.4950	0.0206
122	-7.0300	46.8240	4.8930	0.0193
123	-6.6960	47.2840	4.4180	0.0195
124	-6.7710	47.2220	4.6480	0.0202
125	-6.9420	46.9920	4.6610	0.0212
126	-7.0350	46.8900	4.9940	0.0196
127	-7.1040	47.1650	5.1060	0.0221
128	-7.0750	46.8780	5.3030	0.0187
129	-6.7300	47.2370	4.6580	0.0190
130	-7.0670	46.7770	5.0890	0.0189
131	-6.7140	47.1040	5.1790	0.0151
132	-6.9480	47.0540	5.1660	0.0184
133	-6.9230	46.9910	5.2390	0.0173
134	-6.7590	47.1670	5.2040	0.0162
135	-6.9760	46.8260	5.4060	0.0170
136	-6.6600	47.7220	4.0020	0.0221
137	-6.6300	47.8980	5.1610	0.0187
138	-7.1170	47.0820	5.7310	0.0217
139	-7.2610	47.0100	5.4510	0.0228
140	-7.1050	46.3540	4.9520	0.0181

Conclusion

This aim of this work was never to revise the cosmic distance scale by seeking to determine a Hyades distance radically different from that found by other investigations. Such a purpose would have only led to disappointment, as the distance estimate of 49.7 ± 1.0 parsecs obtained by the least squares with probabilistic constraints is in good agreement with the values found by other astronomers, such as Reid (1992). Rather, this thesis had as its goal the first-time application of a new adjustment technique to a particular problem in kinematic astronomy, one that has piqued the interest of astronomers since the beginning of this century. This procedure is capable of handling all types of kinematic data — proper motions, radial velocities and parallaxes; no information is ignored. It enables the kinematics of the cluster to be incorporated into the adjustment process, thereby eliminating the absurd requirement of a common velocity for all stars which are members of the Hyades. The parallaxes of these stars can also be constrained to prevent the observation errors from distorting the cluster along the line of sight. Finally, it has now been demonstrated that this adjustment procedure yields more accurate velocity and distance estimates than the standard convergent point procedures still used in open cluster investigations.

The only excuses that may be given for not using a stochastically-constrained adjustment to estimate the kinematical properties of open clusters are that it requires a good deal of computing power and considerable time spent in the writing and debugging of computer programs. These excuses seem flimsy in light of the efforts spent designing and building modern astronomical instruments. Shall we spend years in the development and construction of new instrumentation, only to see the data provided by these instruments wasted in adjustments that do not yield the best possible parameter values? One would hope not, as certainly there remains much work to be done before the distances to the Hyades stars can be definitively determined, especially with regard to the biases in the proper motions. It has been remarked that the data provided by the HIPPARCOS astrometric satellite will provide definitive distance estimates of all known Hyades members, which may well prove to be true. Whatever the quality of the HIPPARCOS data, this much is certain; the distances and velocities of the Hyades stars have not, and never will be, definitively established by any convergent point solution. New instruments demand new adjustment techniques.

APPENDIX A
FORTRAN ADJUSTMENT PROGRAM

Note: This program does not check for convergence, but simply performs a set number of iterations, outputting the values of the parameter corrections on each pass.

```
c234567
c     This program does a constrained least square fit to an open
cluster
c
c     dimension and declare arrays
c
      implicit real*8 (a-h,o-z)
      real*8 ks2pyr
c
c     observation vectors
c
      dimension ra(140),dc(140),pmra(140),pmdc(140),rv(140),plx(140)
      dimension epmra(140),epmdc(140),erv(140),eplx(140),nobsv(140)
      dimension vmag(140),blessv(140),bdnum(140)
c
c     initial observation vectors
c
      dimension ra0(140),dc0(140),pmra0(140),pmdc0(140),rv0(140)
      dimension plx0(140)
c
c     major array dimensions
c
      dimension ah(420,4),xh(420,457),sh(457,457),rhoh(560,560)
      dimension w(420,420),atwa(4,4),bh(420,560)
c
c     major vector dimensions
c
      dimension par(4),beta(560),res(457),parcor(4),fh(420),phi(420)
      dimension pstar(140),vxstar(140),vystar(140),vzstar(140)
      dimension respra(140),respdc(140),resrv(140),resplx(140)
      dimension epar(4)
c
c     work array dimensions
c
      dimension work1(457,420),work2(420,420),work3(420,420)
      dimension work4(420,4),work5(560,420)
c
c     work vector dimensions
c
      dimension wvec1(420),wvec2(420),wvec3(4),wvec4(457),wvec5(560)
c
c     establish constants and conversions
c
```

```
      pi=3.14159654d0
      dg2rad=pi/1.8d2
      ks2pyr=1.027712166D-6
      sc2rad=dg2rad/3.6d3
c
c     set up dimensions
c
      n=4
      m=140
      m3=420
      icx=457
      m4=560
c
c     parameter guesses
c
      par(1)=-5.0d0*ks2pyr
      par(2)=4.50d1*ks2pyr
      par(3)=5.5d0*ks2pyr
      par(4)=2.25d-2
c
c     constraints
c
      sxdot=0.25*ks2pyr
      sydot=sxdot
      szdot=sxdot
      splx=2.0d-3
c
c     get data
c
      call datard(bdnum,ra,dc,pmra,epmra,pmdc,epmdc,rv,
     &            erv,plx,eplx,nobsv,vmag,blessv,m)
c
c
c     debug
c
      write (*,1999) (nobsv(i), i=1,40)
 1999 format(40i2)
c
c     initialize observation vector
c
      do 10 i=1,m
         ra0(i)=ra(i)
         dc0(i)=dc(i)
         pmra0(i)=pmra(i)
         pmdc0(i)=pmdc(i)
         rv0(i)=rv(i)
  10  plx0(i)=plx(i)
c
c     initialize all arrays
c
      call clearr(m3,n,ah)
      call clearr(m3,icx,xh)
      call clearr(m3,m4,bh)
      call clearr(m3,m3,w)
      call clearr(icx,icx,sh)
      call clearr(m4,m4,rhoh)
      call clearr(icx,m3,work1)
      call clearr(m3,m3,work2)
      call clearr(m3,m3,work3)
      call clearr(m3,n,work4)
      call clearr(m4,m3,work5)
c
c     initialize vectors
c
      call clearv(icx,res)
      call clearv(m4,beta)
      call clearv(m4,wvec5)
      call clearv(icx,wvec4)
      call clearv(m3,fh)
      call clearv(m3,phi)
      call clearv(m3,wvec1)
      call clearv(m3,wvec2)
      call clearv(n,wvec3)
      call clearv(n,parcor)
c
c     compute sigma and rho covariance matrices
c
      call getsig(epmra,epmdc,erv,eplx,nobsv,m,icx,sh)
      call getrho(sxdot,sydot,szdot,splx,m4,rhoh)
c
c     start loop
c
      do 25 i=1,10
c
c        compute a
c
         call geta(par,ra,dc,m3,n,ah)
c
c        calculate x
c
         call getx(par,ra,dc,m3,icx,nobsv,xh)
         write (*,*) 'a and x computed'
c
c        calculate x * sigma * x transpose
c
         call mxmtg(sh,xh,icx,icx,m3,work1)
         call mxmg(xh,work1,m3,icx,m3,work2)
         write (*,*) 'x * sigma * xt computed'
c
c        compute b
c
         call getb(ah,m3,n,m4,nobsv,bh)
c
c        calculate b * rho * b transpose
c
         call mxmtg(rhoh,bh,m4,m4,m3,work5)
         call mxmg(bh,work5,m3,m4,m3,work3)
         write (*,*) 'b * rho * bt computed'
c
c        add the two
c
         call matadd(work2,work3,m3,m3,w)
c
c        invert the "weight" matrix, w
c
         call spsinv(w,m3)
         write (*,*) 'W inverted'
c
c        compute phi vector
c
```

```
         call mxvg(xh,res,m3,icx,wvec1)
         call getf(ra,dc,par,rv,pmra,pmdc,m3,fh)
         call vsubg(fh,wvec1,m3,phi)
c
c        compute corrections to the parameters
c
         call mxvg(w,phi,m3,m3,wvec2)
         call mtxvg(ah,wvec2,n,m3,wvec3)
c
c           compute (a transpose * w * a) ** -1
c
            call mxmg(w,ah,m3,m3,n,work4)
            call mtxmg(ah,work4,n,m3,n,atwa)
            call syminv(atwa,n)
c
c           bring it all together
c
            call mxvg(atwa,wvec3,n,n,parcor)
            call negate(n,parcor)
c
c        compute the residuals
c
         call mxvg(ah,parcor,m3,n,wvec1)
         call vaddg(phi,wvec1,m3,wvec2)
         call mxvg(w,wvec2,m3,m3,wvec1)
         call mtxvg(xh,wvec1,icx,m3,wvec4)
         call mxvg(sh,wvec4,icx,icx,res)
         call negate(icx,res)
c
c        correct the parameters
c
         call vaddg(par,parcor,n,par)
c
c        correct the observations and separate the residuals
c
         knt=1
         do 20 kk=1,m
            kk1=knt+1
            kk2=knt+2
            kk3=knt+3
c
            resrv(kk)=res(knt)/ks2pyr
            respra(kk)=res(kk1)/sc2rad
            respdc(kk)=res(kk2)/sc2rad
            resplx(kk)=res(kk3)
c
            rv(kk)=rv0(kk)+res(knt)
            pmra(kk)=pmra0(kk)+res(kk1)
            pmdc(kk)=pmdc0(kk)+res(kk2)
            plx(kk)=plx0(kk)+res(kk3)
            knt=knt+nobsv(kk)
  20     continue
c
c     outputs to check for convergence
c
      write (*,600) (parcor(j), j=1,n)
 600  format(4e20.8)
c
  25  continue
c
```

```
c     convergence has been achieved, compute the parameters
c
      vxc=par(1)/ks2pyr
      vyc=par(2)/ks2pyr
      vzc=par(3)/ks2pyr
      plxc=par(4)
c
c     now calculate the beta vector
c
      call mtxvg(bh,wvec1,m4,m3,wvec5)
      call mxvg(rhoh,wvec5,m4,m4,beta)
      call negate(m4,beta)
c
c     compute individual star velocities and parallaxes
c
      do 30 j=1,m4,4
         knt=(j-1)/4+1
         j1=j+1
         j2=j+2
         j3=j+3
         vxstar(knt)=(par(1)+beta(j))/ks2pyr
         vystar(knt)=(par(2)+beta(j1))/ks2pyr
         vzstar(knt)=(par(3)+beta(j2))/ks2pyr
   30 pstar(knt)=par(4)+beta(j3)
c
c     compute errors in the parameters
c
      do 40 j=1,n
         epar(j)=dsqrt(atwa(j,j))
         if (j.lt.4) then
            epar(j)=epar(j)/ks2pyr
         endif
   40 continue
c
c     display the parameters and their errors
c
      write (*,605) vxc,vyc,vzc,plxc
      write (*,605) (epar(j), j=1,4)
  605 format(3f8.2,f7.4)
c
c     open output file
c
      open (4,file='hyadesp.out')
c
c     write results to output file
c
      do 50 j=1,m
   50 write (4,400) resrv(j),respra(j),respdc(j),resplx(j),vxstar(j),
     &              vystar(j),vzstar(j),pstar(j)
  400 format(f7.3,3f9.5,3f8.3,f9.5)
c
c     finished
c
      close (4)
      stop
      end
c234567
      subroutine getx(param,ra,dc,m3,nx,nobsv,x)
c
c     INPUTS:
```

```
c     param is the parameter vector: x dot, y dot, z dot, plx,
respectively
c     ra is the star right ascension vector (radians)
c     dc is the star declination vector
c     m3 is the number of condition equations
c     nx is the column dimension of x = sum of the elements of nobsv
c     nobsv is the vector containing the number of observations for each
c        star (2, 3 or 4)
c
c     OUTPUTS:
c     x is the Jacobian matrix of the partials of the condition
equations
c        with respect to the observations
c
c     declare and dimension variables
c
      implicit real*8 (a-h,o-z)
      dimension param(*),ra(*),dc(*),nobsv(*),x(m3,nx)
c
c     initialize column pointer
c
      jpos=1
c
c     start loop
c
      do 50 i=1,m3,3
         knt=(i-1)/3+1
         i2=i+1
         i3=i+2
         j2=jpos+1
         j3=jpos+3
         j4=jpos+2
         x(i,jpos)=1.0d0
         x(i2,j2)=1.0d0
         x(i3,j4)=1.0d0
         if (nobsv(knt) .gt. 3) then
            x(i2,j3)=param(1)*dsin(ra(knt))-param(2)*dcos(ra(knt))
            x(i3,j3)=param(1)*dcos(ra(knt))*dsin(dc(knt))+
     &               param(2)*dsin(ra(knt))*dsin(dc(knt))-
     &               param(3)*dcos(dc(knt))
         end if
c
c        increment column pointer
c
         jpos=jpos+nobsv(knt)
  50  continue
c
c     finished
c
      end
c
      subroutine geta(param,ra,dc,m3,n,a)
c
c     INPUTS:
c     param is the parameter vector: x dot, y dot, z dot, plx,
respectively
c     ra is the star right ascension vector (radians)
c     dc is the star declination vector
c     m3 is the number of condition equations
c     n is number of unconstrained parameters
```

```
c
c     OUTPUTS:
c     a is the Jacobian matrix of partials of the condition equations
with
c     respect to the parameters
c
c     declare and dimension variables
c
      implicit real*8 (a-h,o-z)
      dimension param(*),ra(*),dc(*),a(m3,n)
c
c     start loop
c
      do 10 i=1,m3,3
         knt=(i-1)/3+1
         i2=i+1
         i3=i+2
         a(i,1)=-dcos(ra(knt))*dcos(dc(knt))
         a(i,2)=-dsin(ra(knt))*dcos(dc(knt))
         a(i,3)=-dsin(dc(knt))
         a(i2,1)=param(4)*dsin(ra(knt))
         a(i2,2)=-param(4)*dcos(ra(knt))
         a(i2,4)=param(1)*dsin(ra(knt))-param(2)*dcos(ra(knt))
         a(i3,1)=param(4)*dcos(ra(knt))*dsin(dc(knt))
         a(i3,2)=param(4)*dsin(ra(knt))*dsin(dc(knt))
         a(i3,3)=-param(4)*dcos(dc(knt))
         a(i3,4)=param(1)*dcos(ra(knt))*dsin(dc(knt))+
     &           param(2)*dsin(ra(knt))*dsin(dc(knt))-
     &           param(3)*dcos(dc(knt))
   10 continue
c
c     finished
c
      return
      end
c
      subroutine getf(ra,dc,param,rv,pmra,pmdc,m3,f)
c
c     INPUTS:
c     param is the parameter vector: x dot, y dot, z dot, plx,
respectively
c     ra is the star right ascension vector (radians)
c     dc is the star declination vector
c     rv is the radial velocity vector
c     pmra is the right ascension proper motion vector
c     pmdc is the declination proper motion vector
c     m3 is the number of condition equations
c
c     OUTPUTS:
c     f is the vector of the condition equations evaluated at the
current
c     values of the observations and the parameters
c
c     declare and dimension variables
c
      implicit real*8 (a-h,o-z)
      dimension param(*),ra(*),dc(*),rv(*),pmra(*),pmdc(*),f(m3)
c
c     start loop
c
```

```
      do 10 i=1,m3,3
         knt=(i-1)/3+1
         i2=i+1
         i3=i+2
         f(i)=rv(knt)-param(1)*dcos(ra(knt))*dcos(dc(knt))
     &          -param(2)*dsin(ra(knt))*dcos(dc(knt))
     &          -param(3)*dsin(dc(knt))
         f(i2)=pmra(knt)+param(4)*(param(1)*dsin(ra(knt))
     &           -param(2)*dcos(ra(knt)))
         f(i3)=pmdc(knt)+param(4)*(param(1)*dcos(ra(knt))*dsin(dc(knt))
     &           +param(2)*dsin(ra(knt))*dsin(dc(knt))
     &           -param(3)*dcos(dc(knt)))
   10 continue
c
c     finished
c
      return
      end
c
      subroutine getsig(epmra,epmdc,erv,eplx,nobsv,m,ms,sigma)
c
c     INPUTS:
c     epmra is the vector of errors in the ra proper motions
c     epmdc is the vector of errors in the dec proper motions
c     erv is the vector of errors in the radial velocities
c     eplx is the vector of errors in the parallaxes
c     m is the number of observations
c     ms is the number of rows (or columns) of sigma
c
c     OUTPUTS:
c     sigma is the covariance matrix of the observations
c
c     declare and dimension variables
c
      implicit real*8 (a-h,o-z)
      dimension epmra(*),epmdc(*),erv(*),eplx(*),nobsv(*)
      dimension sigma(ms,ms)
c
c     start loop
c
      jpos=1
      do 10 i=1,m
         j2=jpos+1
         j3=jpos+2
         j4=jpos+3
         sigma(jpos,jpos)=erv(i)*erv(i)
         sigma(j2,j2)=epmra(i)*epmra(i)
         sigma(j3,j3)=epmdc(i)*epmdc(i)
         sigma(j4,j4)=eplx(i)*eplx(i)
         jpos=jpos+nobsv(i)
   10 continue
      return
      end
c
      subroutine getrho(sxdot,sydot,szdot,splx,m4,rho)
c
c     INPUTS:
c     sxdot is the velocity constraint on the x component
c     sydot is the velocity constraint on the y component
c     szdot is the velocity constraint on the z component
```

```
c     splx is the parallax constraint, obtained from cluster angular
c          diameter
c     m4 is the row or column dimension of rho
c
c     OUTPUTS:
c     rho is the covariance matrix of the constrained parameters
c
c     declare and dimension variables
c
      implicit real*8 (a-h,o-z)
      dimension rho(m4,m4)
c
c     start loop
c
      do 10 i=1,m4,4
         i2=i+1
         i3=i+2
         i4=i+3
         rho(i,i)=sxdot*sxdot
         rho(i2,i2)=sydot*sydot
         rho(i3,i3)=szdot*szdot
         rho(i4,i4)=splx*splx
   10 continue
c
c     finished
c
      return
      end
c
      subroutine getb(a,m3,n,m4,nobsv,b)
c
c     INPUTS:
c     a is the jacobian matrix of partials with respect to the
parameters
c     m3, n are the row and column dimensions of a
c     m4 is the column dimension of b
c     nobsv is the vector containing the number of observations for each
c        star (2, 3 or 4)
c
c     OUTPUTS:
c     b is the Jacobian matrix of the condition equations with respect
to the
c        constrained parameters
c
c     declare and dimension variables
c
      implicit real*8 (a-h,o-z)
      dimension a(m3,n),nobsv(*),b(m3,m4)
c
c     start loop
c
      j=0
      do 20 i=1,m3,3
         i2=i+1
         i3=i+2
         do 10 k=1,4
            ic=j+k
            b(i,ic)=a(i,k)
            b(i2,ic)=a(i2,k)
            b(i3,ic)=a(i3,k)
```

```
  10     continue
         j=j+4
  20  continue
c
c     finished
c
      return
      end
c
      subroutine getdta(ra,dc,pmra,epmra,pmdc,epmdc,rv,erv,plx,eplx,
     &                  nobsv,m)
c
c     INPUTS:
c     m is the number of stars
c
c     OUTPUTS:
c     ra is the vector of star ra's, units are radians
c     dc is the vector of star declinations, units are radians
c     pmra is the vector of star ra proper motions, units are radians
c     epmra is the vector of standard errors associated with pmra
c     pmdc is the vector of star declination proper motions, units are
radians
c     epmdc is the vector of standard errors associated with pmdc
c     rv is the vector of star radial velocities, units are pc per yr
c     erv is the vector of standard errors associated with rv
c     plx is the vector of star parallaxes, units are seconds
c     eplx is the vector of standard errors associated with plx
c     nobsv is the vector containing the number of observations for each
c        star (2, 3 or 4)
c
c     declare and dimension variables
c
      implicit real*8 (a-h,o-z)
      real*8 ks2pyr
      dimension ra(*),dc(*),pmra(*),pmdc(*),rv(*),plx(*)
      dimension epmra(*),epmdc(*),erv(*),eplx(*),nobsv(*)
c
c     establish constants and conversions
c
      pi=3.14159654d0
      dg2rad=pi/1.8d2
      ks2pyr=1.027712166D-6
      sc2rad=dg2rad/3.6d3
c
c     open file
c
      open (4, file='stars.in')
c
c     start loop
c
      do 10 i=1,m
         read (4,400,end=90) alpha,delta,pra,epra,pdc,epdc,plx(i),
     &                       eplx(i),rho,erho,nobsv(i)
 400     format(3x,2f11.6,6f9.5,2f6.2,i2)
c
c        convert to the appropriate units
c
         ra(i)=alpha*dg2rad
         dc(i)=delta*dg2rad
         pmra(i)=pra*sc2rad
         epmra(i)=epra*sc2rad
         pmdc(i)=pdc*sc2rad
         epmdc(i)=epdc*sc2rad
         rv(i)=rho*ks2pyr
         erv(i)=erho*ks2pyr
   10 continue
c
c     finished
c
   90 close (4)
      return
      end
c
      subroutine spsinv(array,m3)
c
c     INPUTS:
c     array is a square sparse matrix of dimension m3 with 3x3 blocks
c     m3 is dimension of array
c
c     OUTPUTS:
c     array is returned as its inverse (inverted in place)
c
c     dimension and declare variables
c
      implicit real*8 (a-h,o-z)
      dimension array(m3,m3), temp(3,3)
c
c     start outer loop
c
      jpos=0
      do 20 i=1,m3,3
         i2=i+1
         i3=i+2
c
c        put block into temporary array
c
         do 5 j=1,3
            j1=j+jpos
            temp(1,j)=array(i,j1)
            temp(2,j)=array(i2,j1)
    5    temp(3,j)=array(i3,j1)
c
c        invert temporary array
c
         call syminv(temp,3)
c
c        put inverted block into place
c
         do 10 j=1,3
            j1=j+jpos
            array(i,j1)=temp(1,j)
            array(i2,j1)=temp(2,j)
   10    array(i3,j1)=temp(3,j)
c
c        increment column
c
         jpos=jpos+3
c
   20 continue
c
```

```
c     finished
c
      return
      end
       subroutine effcal(evec,cvec,n,effcy)
c
c     this subroutine will calculate the 'efficiency' of a fit, given
c     the vector of the diagonal of the scaled covariance matrix, and
c     its associated eigenvalues
c
      implicit real*8 (a-h,o-z)
      dimension evec(n),cvec(n)
c
c     calculate the product of the eigenvalues
c
      eprd=1.0d0
      do 10 i=1,n
   10 eprd=evec(i)*eprd
c
c     calculate the product of the covariance diagonal elements
c
      cprd=1.0d0
      do 20 i=1,n
   20 cprd=cvec(i)*cprd
c
c     the efficiency is now equal to eprd/cprd to the 1/nth power
c
      epwr=1.0d0/n
      effcy=(eprd/cprd)**epwr
c
c     finished
c
      return
      end
c234567
      subroutine clearv(ndim,vector)
c
c     this subroutine will fill a vector with zeros
c
      implicit real*8 (a-h,o-z)
      dimension vector(*)
c
c     fill vector with 0's
c
      do 10 i=1,ndim
   10 vector(i)=0.0d0
c
c     finished
c
      return
      end
c
c234567
      subroutine clearr(nrow,ncol,array)
c
c     this subroutine will fill an array with zeros
c
      implicit real*8 (a-h,o-z)
      dimension array(nrow,ncol)
c
```

```
c     fill array with 0's
c
      do 10 i=1,nrow
         do 10 j=1,ncol
   10 array(i,j)=0.0d0
c
c     finished
c
      return
      end
c
      SUBROUTINE VTXVG (V1,V2,NDIM,VALUE)
C     This subroutine multiplies a row vector by a column vector,
C     outputting a scalar, VALUE
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION V1(*),V2(*)
      VALUE=0.0D0
      DO 10 I=1,NDIM
   10 VALUE=V1(I)*V2(I)+VALUE
      RETURN
      END
C
      SUBROUTINE MATADD (ARRAY1,ARRAY2,NROW,NCOL,ARRAYO)
C     This subroutine will add 2 double precision arrays of
C     arbitrary size
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION ARRAY1(NROW,NCOL),ARRAY2(NROW,NCOL)
      DIMENSION ARRAYO(NROW,NCOL)
      DO 10 I=1,NROW
         DO 10 J=1,NCOL
   10 ARRAYO(I,J)=ARRAY1(I,J)+ARRAY2(I,J)
      RETURN
      END
C
      SUBROUTINE MATSUB (ARRAY1,ARRAY2,NROW,NCOL,ARRAYO)
C     This subroutine will subtract two double precision arrays
C     of arbitrary size
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION ARRAY1(NROW,NCOL),ARRAY2(NROW,NCOL)
      DIMENSION ARRAYO(NROW,NCOL)
      DO 10 I=1,NROW
         DO 10 J=1,NCOL
   10 ARRAYO(I,J)=ARRAY1(I,J)-ARRAY2(I,J)
      RETURN
      END
C
      SUBROUTINE NEGATE (NROW,VECTOR)
C
C     This subroutine returns the negative of VECTOR. VECTOR is replaced
C     by its negative.
C
C     Dimension array
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION VECTOR(*)
C     Negate VECTOR
      DO 10 I=1,NROW
   10 VECTOR(I)=-1.0*VECTOR(I)
C     Finished
      RETURN
      END
```

```
c
      SUBROUTINE MTXMG (M1, M2, NC1, NR1R2, NC2, MOUT)
      DOUBLE PRECISION M1(NR1R2,NC1), M2(NR1R2,NC2)
      INTEGER*4 NC1, NR1R2, NC2
      DOUBLE PRECISION MOUT(NC1,NC2)
C
C  Multiply the transpose of a matrix and a matrix, of arbitrary size.
C
C  VARIABLE  I/O              DESCRIPTION
C   M1        I      Left-hand matrix whose transpose is to be multiplied
C   M2        I      Right-hand matrix to be multiplied.
C   NC1       I      Column dimension of M1 and row dimension of MOUT.
C   NR1R2     I      Row dimension of M1 and row dimension of M2.
C   NC2       I      Column dimension of M2 and column dimension of MOUT.
C   MOUT      O      Product matrix M1**T * M2.
C                    MOUT must not overwrite either M1 or M2.
C
C  VERSION 1.00 -- 1984 Nov 08 -- W.M. Owen
C
C-
      DOUBLE PRECISION SUM
      INTEGER I,J,K
C
C  Perform the matrix multiplication
C
      DO 10 I=1,NC1
         DO 10 J=1,NC2
            SUM = 0.D0
            DO 5 K=1,NR1R2
    5          SUM = SUM + M1(K,I)*M2(K,J)
   10       MOUT(I,J) = SUM
C
      RETURN
      END
c
      SUBROUTINE MTXVG (M1, V2, NC1, NR1R2, VOUT)
      DOUBLE PRECISION M1(NR1R2,NC1), V2(NR1R2)
      INTEGER*4 NC1, NR1R2
      DOUBLE PRECISION VOUT(NC1)
C
C  Multiply the transpose of a matrix and a vector of arbitrary size.
C
C  VARIABLE  I/O              DESCRIPTION
C   M1        I      Left-hand matrix whose transpose is to be multiplied
C   V2        I      Right-hand vector to be multiplied.
C   NC1       I      Column dimension of M1 and length of VOUT.
C   NR1R2     I      Row dimension of M1 and length of V2.
C   VOUT      O      Product vector M1**T * V2.
C                    VOUT must not overwrite either M1 or V2.
C
C  VERSION 1.00 -- 1984 Nov 08 -- W.M. Owen
C
C-
      DOUBLE PRECISION SUM
      INTEGER I,K
C
C  Perform the matrix-vector multiplication
C
      DO 10 I=1,NC1
         SUM = 0.D0
         DO 5 K=1,NR1R2
    5       SUM = SUM + M1(K,I)*V2(K)
   10    VOUT(I) = SUM
C
      RETURN
      END
C
      SUBROUTINE MXMG (M1, M2, NR1, NC1R2, NC2, MOUT)
      DOUBLE PRECISION M1(NR1,NC1R2), M2(NC1R2,NC2)
      INTEGER*4 NR1, NC1R2, NC2
      DOUBLE PRECISION MOUT(NR1,NC2)
C
C  Multiply two matrices of arbitrary size.
C
C  VARIABLE  I/O              DESCRIPTION
C   M1        I      Left-hand matrix to be multiplied.
C   M2        I      Right-hand matrix to be multiplied.
C   NR1       I      Row dimension of M1 (and also MOUT).
C   NC1R2     I      Column dimension of M1 and row dimension of M2.
C   NC2       I      Column dimension of M2 (and also MOUT).
C   MOUT      O      Product matrix M1*M2.
C                    MOUT must not overwrite either M1 or M2.
C
C  VERSION 1.00 -- 1984 Nov 08 -- W.M. Owen
C
C-
      DOUBLE PRECISION SUM
      INTEGER I,J,K
C
C  Perform the matrix multiplication
C
      DO 10 I=1,NR1
         DO 10 J=1,NC2
            SUM = 0.D0
            DO 5 K=1,NC1R2
    5          SUM = SUM + M1(I,K)*M2(K,J)
   10       MOUT(I,J) = SUM
C
      RETURN
      END
C
      SUBROUTINE MXMTG (M1, M2, NR1, NC1C2, NR2, MOUT)
      DOUBLE PRECISION M1(NR1,NC1C2), M2(NR2,NC1C2)
      INTEGER*4 NR1, NC1C2, NR2
      DOUBLE PRECISION MOUT(NR1,NR2)
C
C  Multiply a matrix and the transpose of a matrix, of arbitrary size.
C
C  VARIABLE  I/O              DESCRIPTION
C   M1        I      Left-hand matrix to be multiplied.
C   M2        I      Right-hand matrix whose transpose is to be mult.
C   NR1       I      Row dimension of M1 and row dimension of MOUT.
C   NC1C2     I      Column dimension of M1 and column dimension of M2.
C   NR2       I      Row dimension of M2 and column dimension of MOUT.
C   MOUT      O      Product matrix M1 * M2**T.
C                    MOUT must not overwrite either M1 or M2.
C
C  VERSION 1.01 -- 1984 Nov 12 -- W.M. Owen
C
C-
```

```
      DOUBLE PRECISION SUM
      INTEGER I,J,K
C
C  Perform the matrix multiplication
C
      DO 10 I=1,NR1
         DO 10 J=1,NR2
            SUM = 0.D0
            DO 5 K=1,NC1C2
    5          SUM = SUM + M1(I,K)*M2(J,K)
   10       MOUT(I,J) = SUM
C
      RETURN
      END
c
      SUBROUTINE MXVG (M1, V2, NR1, NC1R2, VOUT)
      DOUBLE PRECISION M1(NR1,NC1R2), V2(NC1R2)
      INTEGER*4 NR1, NC1R2
      DOUBLE PRECISION VOUT(NR1)
C
C  Multiply a matrix and a vector of arbitrary size.
C
C  VARIABLE  I/O              DESCRIPTION
C   M1        I     Left-hand matrix to be multiplied.
C   V2        I     Right-hand vector to be multiplied.
C   NR1       I     Row dimension of M1 and length of VOUT.
C   NC1R2     I     Column dimension of M1 and length of V2.
C   VOUT      O     Product vector M1*V2.
C                   VOUT must not overwrite either M1 or V2.
C
C  VERSION 1.00 -- 1984 Nov 08 -- W.M. Owen
C
C-
      DOUBLE PRECISION SUM
      INTEGER I,K
C
C  Perform the matrix-vector multiplication
C
      DO 10 I=1,NR1
         SUM = 0.D0
         DO 5 K=1,NC1R2
    5       SUM = SUM + M1(I,K)*V2(K)
   10    VOUT(I) = SUM
C
      RETURN
      END
c
      SUBROUTINE VADDG (V1, V2, NDIM, VOUT)
      DOUBLE PRECISION V1(NDIM), V2(NDIM)
      INTEGER*4 NDIM
      DOUBLE PRECISION VOUT(NDIM)
C
C  Add two vectors of arbitrary size.
C
C  VARIABLE  I/O              DESCRIPTION
C   V1        I     First vector to be added.
C   V2        I     Second vector to be added.
C   NDIM      I     Dimension of V1, V2, and VOUT.
C   VOUT      O     Sum vector, V1 + V2.
C                   VOUT can overwrite either V1 or V2.
```

```
C
C  VERSION 1.00 -- 1984 Nov 08 -- W.M. Owen
C
C-
      INTEGER I
C
      DO 1 I=1,NDIM
    1     VOUT(I) = V1(I) + V2(I)
C
      RETURN
      END
c
      SUBROUTINE VSUBG (V1, V2, NDIM, VOUT)
      DOUBLE PRECISION V1(NDIM), V2(NDIM)
      INTEGER*4 NDIM
      DOUBLE PRECISION VOUT(NDIM)
C
C  Subtract two vectors of arbitrary size.
C
C  VARIABLE  I/O              DESCRIPTION
C   V1        I     First vector (minuend).
C   V2        I     Second vector (subtrahend).
C   NDIM      I     Dimension of V1, V2, and VOUT.
C   VOUT      O     Difference vector, V1 - V2.
C                   VOUT can overwrite either V1 or V2.
C
C  VERSION 1.00 -- 1984 Nov 08 -- W.M. Owen
C
C-
      INTEGER I
C
      DO 1 I=1,NDIM
    1     VOUT(I) = V1(I) - V2(I)
C
      RETURN
      END
c
      SUBROUTINE SYMINV(ARRAY,NDIM)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION ARRAY(NDIM,NDIM),S(50,50),ARYINV(50,50)
C     CALCULATE S(1,1),S(1,J)
      S(1,1)=DSQRT(ARRAY(1,1))
      DO 10 J=2,NDIM
   10 S(1,J)=ARRAY(1,J)/S(1,1)
C     CALCULATE S(I,I),S(I,J)
      I=2
      SUM3=0.0D0
   15 SUM2=0.0D0
      LOOPLM=I-1
      DO 20 L=1,LOOPLM
   20 SUM2=S(L,I)*S(L,I)+SUM2
      S(I,I)=DSQRT(ARRAY(I,I)-SUM2)
      J=I+1
   25 DO 30 L=1,LOOPLM
   30 SUM3=S(L,I)*S(L,J)+SUM3
      S(I,J)=(ARRAY(I,J)-SUM3)/S(I,I)
      J=J+1
      SUM3=0.0D0
      IF (J .LE. NDIM) GOTO 25
      I=I+1
      IF (I .LE. NDIM) GOTO 15
C     CALCULATE INVERSE
      CD=0.0D0
      SUM=0.0D0
      ARYINV(NDIM,NDIM)=1.0D0/(S(NDIM,NDIM)*S(NDIM,NDIM))
      I=NDIM-1
   40 K=NDIM
   45 IF (K .EQ. I) CD=1.0D0/S(I,I)
      IF (K .NE. I) CD=0.0D0
      LOOPST=I+1
      DO 50 NU=LOOPST,NDIM
   50 SUM=S(I,NU)*ARYINV(NU,K)+SUM
      ARYINV(I,K)=(CD-SUM)/S(I,I)
      ARYINV(K,I)=ARYINV(I,K)
      SUM=0.0D0
      K=K-1
      IF (K .GE. I) GOTO 45
      I=I-1
      IF (I .GE. 1) GOTO 40
      DO 55 I2=1,NDIM
         DO 55 J2=1,NDIM
   55 ARRAY(I2,J2)=ARYINV(I2,J2)
      RETURN
      END
c
c234567
      subroutine jacobi(a,n,np,d,v,nrot)
c
c     this subroutine computes the eigenvectors and eigenvalues of a
real
c     symmetric matrix a, of size n by n, stored in a physical np by np
c     array. On output the elements of a above the diagonal are
destroyed.
c     d returns the eigenvalues of a in its first n elements. v is a
matrix
c     with the same logical and physical dimensions as a whose columns
c     contain, on output, the normalized eigenvectors of a. nrot returns
c     the number of required jacobi rotations.
c
c     this program assumes underflows are treated as zeros!!!!!
c
      implicit real*8 (a-h,o-z)
      parameter (nmax=100)
      dimension a(np,np),d(np),v(np,np),b(nmax),z(nmax)
      do 12 ip=1,n
         do 11 iq=1,n
            v(ip,iq)=0.0d0
   11    continue
         v(ip,ip)=1.0d0
   12 continue
      do 13 ip=1,n
         b(ip)=a(ip,ip)
         d(ip)=b(ip)
         z(ip)=0.0d0
   13 continue
      nrot=0
      do 24 i=1,50
         sm=0.0d0
         do 15 ip=1,n-1
            do 14 iq=ip+1,n
            sm=sm+dabs(a(ip,iq))
14       continue
15    continue
      if (sm.eq.0.0d0) return
      if (i.lt.4) then
         tresh=0.2*sm/n**2
      else
         tresh=0.0d0
      endif
      do 22 ip=1,n-1
         do 21 iq=ip+1,n
            g=1.0d2*dabs(a(ip,iq))
            if ((i.gt.4).and.(dabs(d(ip))+g.eq.dabs(d(ip)))
  &            .and.(dabs(d(iq))+g.eq.dabs(d(iq)))) then
               a(ip,iq)=0.0d0
            else if (dabs(a(ip,iq)).gt.tresh) then
               h=d(iq)-d(ip)
               if (dabs(h)+g.eq.dabs(h)) then
                  t=a(ip,iq)/h
               else
                  theta=0.5*h/a(ip,iq)
                  t=1.0d0/(dabs(theta)+dsqrt(1.0d0+theta**2))
                  if (theta.lt.0.0d0) t=-t
               endif
               c=1.0d0/dsqrt(1+t**2)
               s=t*c
               tau=s/(1.0d0+c)
               h=t*a(ip,iq)
               z(ip)=z(ip)-h
               z(iq)=z(iq)+h
               d(ip)=d(ip)-h
               d(iq)=d(iq)+h
               a(ip,iq)=0.0d0
               do 16 j=1,ip-1
                  g=a(j,ip)
                  h=a(j,iq)
                  a(j,ip)=g-s*(h+g*tau)
                  a(j,iq)=h+s*(g-h*tau)
16             continue
               do 17 j=ip+1,iq-1
                  g=a(ip,j)
                  h=a(j,iq)
                  a(ip,j)=g-s*(h+g*tau)
                  a(j,iq)=h+s*(g-h*tau)
17             continue
               do 18 j=iq+1,n
                  g=a(ip,j)
                  h=a(iq,j)
                  a(ip,j)=g-s*(h+g*tau)
                  a(iq,j)=h+s*(g-h*tau)
18             continue
               do 19 j=1,n
                  g=v(j,ip)
                  h=v(j,iq)
                  v(j,ip)=g-s*(h+g*tau)
                  v(j,iq)=h+s*(g-h*tau)
19             continue
               nrot=nrot+1
            endif
21       continue
```

```
   22    continue
         do 23 ip=1,n
            b(ip)=b(ip)+z(ip)
            d(ip)=b(ip)
            z(ip)=0.0d0
   23    continue
   24 continue
      write(*,*) '50 iterations should never happen!'
      return
      end
c
      subroutine eigsrt(d,v,n,np)
c
c     this subroutine sorts the eigenvalues, d, and the eigenvectors, v,
c     as output by the jacobi routine. It will put the eigenvalues into
c     ascending order, and then rearrange the eigenvectors
correspondingly.
c
      implicit real*8 (a-h,o-z)
      dimension d(np),v(np,np)
      do 13 i=1,n-1
         k=i
         p=d(i)
         do 11 j=i+1,n
            if (d(j).ge.p) then
               k=j
               p=d(j)
            endif
   11    continue
         if (k.ne.i) then
            d(k)=d(i)
            d(i)=p
            do 12 j=1,n
               p=v(j,i)
               v(j,i)=v(j,k)
               v(j,k)=p
   12       continue
         endif
   13 continue
      return
      end
c234567
      subroutine datard(bdnum,ra,dc,pmra,epmra,pmdc,epmdc,rv,
     &                  erv,plx,eplx,nobsv,vmag,blessv,m)
c
c     this subroutine reads the data file schwan.dat, converting
c     everything to the appropriate units...
c
      implicit real*8 (a-h,o-z)
      dimension bdnum(*),ra(*),dc(*),pmra(*),pmdc(*),rv(*)
      dimension epmra(*),epmdc(*),erv(*),vmag(*),blessv(*)
      dimension nobsv(*),plx(*),eplx(*)
      character*1 indrv
      character*7 hdnum
c
c     establish pi, and seconds to radians, km/s to pc/yr conversions
c
      pi=3.141592654d0
      sc2rad=pi/(1.8d2*3.6d3)
      rks2py=1.022712166d-6
```

```
c
c     open data file
c
      open (1,file='schwan2.dat')
c
c     read data
c
      knt=1
    5 read(1,100,end=99) ivbnum,bdnum(knt),hdnum,ra(knt),dc(knt),
     &                   ipmra,iepmra,ipmdc,iepmdc,rv(knt),indrv,
     &                   vmag(knt),blessv(knt),plx(knt),ieplx
c
c     convert everything to the appropriate units...
c       first the proper motion errors to radians/year
c
      if (iepmra.eq.0) iepmra=1
      if (iepmdc.eq.0) iepmdc=1
      epmra(knt)=iepmra*sc2rad/1.0d4
      epmdc(knt)=iepmdc*sc2rad/1.0d4
c
c       get radial velocity error, convert to pc/yr
      if (indrv.eq.'G') erv(knt)=0.5*rks2py
      if (indrv.eq.'W') erv(knt)=5.0*rks2py
c
c     correct the radial velocities for zero point error...
c
      rv(knt)=rv(knt)*rks2py
c
c     convert the proper motions to radians per year...
c
      pmra(knt)=ipmra/1.0d4*sc2rad
      pmdc(knt)=ipmdc/1.0d4*sc2rad
c
c     get the parallax errors
c
      eplx(knt)=ieplx/1.0d4
c
c     store number of observations for this star
c
      nobsv(knt)=3
      if (ieplx .gt. 0) nobsv(knt)=4
c
c     everything is now in the right units, so increment
c     the counter and get another star.
c
      knt=knt+1
      goto 5
c
c     format statements
c
  100 format(i3,f8.3,a7,2f12.8,1x,i4,i3,1x,i4,i3,f5.1,1x,a1,f6.2,f5.2,
     &       f8.4,i4)
c
c     finished, establish number of stars.
c
   99 m=knt-1
      close(1)
      return
      end
```

APPENDIX B
TABLE OF ADJUSTMENT RESIDUALS

Table B-1. Observation residuals.

No.	v_{vr} (km s^{-1})	$v_{\mu\alpha}$ (" yr^{-1})	$v_{\mu\delta}$ (" yr^{-1})	v_{π} (arcsec.)
1	0.1110	0.0000	0.0000	0.0080
2	-0.0180	0.0000	-2.0000e-5	0.0002
3	0.3230	0.0000	1.0000e-5	0.0096
4	0.5660	1.0000e-5	-2.0000e-5	-0.0101
5	-0.7360	-0.0001	-2.0000e-5	
6	1.0890	3.0000e-5	0.0003	0.0184
7	3.5680	4.0000e-5	0.0003	0.0045
8	3.7390	0.0000	-4.0000e-5	-0.0097
9	-1.6300	0.0004	0.0002	
10	7.3640	1.0000e-5	-0.0002	-0.0258
11	2.2500	-0.0001	0.0006	0.0377
12	3.1250	0.0001	0.0001	
13	4.0410	-0.0002	-0.0005	0.0100
14	7.2350	-6.0000e-5	-0.0006	
15	-3.1980	-0.0003	-0.0018	0.0063
16	5.6460	0.0002	0.0006	
17	4.5900	-0.0001	-0.0015	
18	1.7510	0.0001	-0.0029	-0.0409
19	-2.1630	-0.0008	-0.0001	
20	2.1530	0.0002	0.0011	
21	-2.2790	-0.0004	-0.0009	
22	3.4160	-9.0000e-5	-0.0008	
23	2.7620	0.0003	0.0011	
24	-1.2620	0.0002	0.0020	
25	6.6300	-0.0050	-0.0045	
26	4.7360	-0.0008	-0.0049	
27	2.2690	0.0001	0.0006	0.0044
28	3.0370	-6.0000e-5	-7.0000e-5	
29	-0.8880	-0.0007	-0.0016	0.0030
30	7.1150	-0.0007	-0.0011	
31	-0.1520	0.0001	0.0014	0.0022
32	3.6840	0.0002	0.0009	-0.0078
33	3.3430	4.0000e-5	-1.0000e-5	
34	0.1270	0.0002	0.0011	0.0045
35	0.9590	-6.0000e-5	-0.0003	0.0068
36	3.0950	-0.0003	-0.0014	0.0035
37	1.3840	-3.0000e-5	-0.0002	
38	2.0520	-0.0003	-0.0005	
39	4.7420	0.0003	0.0017	

Table B-1 -- continued

No.	ν_{vr} (km s^{-1})	$\nu_{\mu\alpha}$ ('' yr^{-1})	$\nu_{\mu\delta}$ ('' yr^{-1})	ν_{π} (arcsec.)
40	-0.2640	2.0000e-5	0.0012	
41	2.9890	0.0004	0.0014	
42	0.2380	0.0004	0.0015	
43	3.0100	-0.0002	-0.0005	
44	-0.6800	4.0000e-5	0.0006	
45	-0.0640	-0.0032	-0.0050	
46	-0.0250	-0.0012	6.0000e-5	
47	-0.4030	-0.0012	-0.0044	
48	-0.2030	-0.0013	0.0003	
49	-0.0500	-0.0006	-0.0016	
50	-0.0920	0.0002	0.0006	
51	1.8790	-0.0004	-0.0011	
52	-0.0250	0.0006	0.0007	
53	4.4300	0.0004	2.0000e-5	
54	0.1390	-0.0004	0.0003	
55	0.3260	-0.0013	-0.0031	
56	-0.1480	3.0000e-5	-0.0007	
57	-0.2090	-0.0010	-0.0028	
58	-3.1370	-0.0016	-0.0021	
59	-0.4130	0.0021	-0.0034	
60	-0.0620	2.0000e-5	-2.0000e-5	
61	0.1760	-0.0024	-0.0019	
62	-1.6300	0.0030	-0.0039	
63	-0.2020	-0.0004	0.0004	
64	-0.0380	-0.0003	-0.0004	
65	0.8640	-0.0027	0.0017	
66	5.0850	-0.0010	-0.0028	
67	-0.1000	5.0000e-5	0.0024	
68	-0.0720	-0.0005	-0.0041	
69	0.3710	0.0002	0.0037	
70	0.4670	0.0001	0.0001	
71	-4.8990	-0.0002	-0.0007	
72	-0.1430	-0.0002	0.0006	
73	-0.3450	0.0002	0.0020	
74	0.2200	-0.00010	-0.0041	
75	0.1220	-0.0003	0.0013	
76	0.1680	-0.0004	-0.0004	
77	-2.7270	-0.0009	-0.0021	0.0064
78	2.3620	-0.0014	-0.0043	
79	0.3160	-0.0004	-0.0024	
80	-0.3270	-0.0002	0.0019	
81	-1.3810	-0.0038	-0.0052	
82	-0.2360	0.0022	0.0050	
83	0.3050	0.0001	0.0075	0.0236
84	-0.1320	-0.0008	-0.0020	
85	1.2870	-0.0005	-0.0011	0.0037
86	0.0690	0.0009	-0.0001	
87	-3.2740	0.0011	0.0011	
88	0.5500	-0.0028	-0.0041	
89	-0.4420	-0.0019	-0.0034	

Table B-1 -- continued

No.	ν_{vr} (km s^{-1})	$\nu_{\mu\alpha}$ ('' yr^{-1})	$\nu_{\mu\delta}$ ('' yr^{-1})	ν_{π} (arcsec.)
90	0.1160	0.0004	0.0021	
91	-1.1300	-0.0013	-0.0017	0.0054
92	0.3590	-0.0009	-0.0027	0.0079
93	-0.0170	-0.0003	-0.0003	0.0070
94	-0.2240	0.0002	0.0008	
95	0.5400	7.0000e-5	0.0005	0.0095
96	0.2520	-0.0001	-0.0011	
97	-0.4950	0.0019	0.0019	
98	-0.1490	0.0005	-0.0044	
99	-3.5890	-0.0004	-0.0015	
100	0.0330	-2.0000e-5	-0.0010	-0.0012
101	0.1620	-0.0001	-0.0003	0.0166
102	-4.4540	0.0002	-0.0018	
103	0.2240	0.0004	0.0020	
104	0.8560	0.0004	0.0018	0.0073
105	-0.1850	-0.0003	-0.0002	
106	-0.2370	-3.0000e-5	-0.0001	
107	2.0950	-0.0003	-0.0014	
108	-0.0530	0.0005	0.0002	-0.0031
109	-2.8890	0.0001	0.0006	
110	-0.1480	0.0008	0.0014	-0.0031
111	-2.0710	0.0026	0.0082	-0.0131
112	-0.3300	0.0008	0.0015	
113	-0.2020	0.0034	0.0041	-0.0144
114	6.1590	-0.0002	-0.0026	-0.0057
115	3.4520	0.0008	0.0025	
116	-0.0030	0.0015	0.0014	
117	0.1640	0.0011	0.0036	
118	-0.1160	0.0039	0.0078	-0.0060
119	7.4920	0.0018	0.0038	
120	-0.0940	0.0003	0.0012	
121	-0.2930	-0.0003	-0.0010	
122	0.3470	0.0026	0.0050	
123	-0.0490	0.0012	0.0036	
124	0.0470	0.0010	0.0028	
125	0.2570	0.0006	0.0019	
126	0.2630	0.0004	0.0007	
127	0.7330	-0.0004	0.0007	
128	0.2050	0.0002	-0.0005	
129	-0.0460	0.0018	0.0036	
130	0.3440	0.0018	0.0013	
131	-0.0740	0.0017	0.0004	
132	4.1070	0.0024	0.0011	
133	7.3900	0.0026	-0.0005	
134	-0.1200	0.0011	0.0002	-0.0082
135	0.2090	0.0031	-0.0042	
136	-0.5410	-0.0005	0.0054	0.0068
137	-0.8690	0.0008	0.0019	
138	0.9630	-0.0003	-0.0027	
139	0.1310	-0.0010	-0.0012	
140	0.7690	0.0028	0.0040	

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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